Areas of Shapes between Parallel Lines

Overview

In this unit, students will discuss the concept of minimum information needed to construct geometric objects. They will also observe the relationship between area, the base and the height of a parallelogram. They will further investigate other shapes between two parallel lines.

Type of Learning Unit: Classroom Activity

Minimum Time Required: 2 sessions of 40 minutes each

The task/objective of the activity

I. To know the minimum data required to draw a line parallel to a given line.
II. To observe the relation of area of a parallelogram with base, height and perimeter.
III. To understand area of shapes between two parallel lines with common base.

Prerequisite concepts that must be known to the student

Basic Constructions, Formula of area of a parallelogram and trapezium

Materials/facilities Required: Sheets of plain paper and square grid paper, pencil, scale, compass

Introduction

In this learning unit, we will explore some properties of shapes between parallel lines.

Task 1

1.Draw a line *l* and choose a point P not on *l*.

a) How many lines can pass through P? ______________________

b) Out of these how many lines will be (i) parallel to *l* ? (ii) perpendicular to *l* ?

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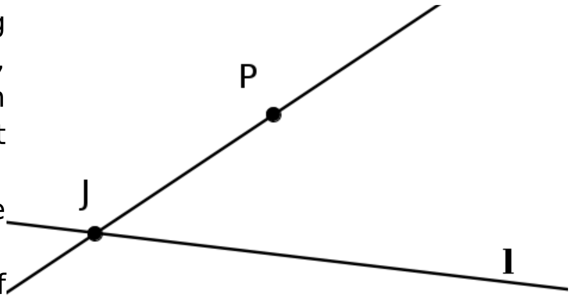
2. Draw a line *l*' through P parallel to *l*.

Given *l* and P, what are the steps required to construct *l*'?

Compare your steps of construction with those of your friends. Why do you think these steps give you a parallel line? Justify the steps of construction.

One way of doing this construction is by drawing an angle with one of its arms and its vertex, say J, on l and P is on the other arm, and then constructing an angle of equal measure such that the vertex is at P and one of the arms is along PJ.
The other arm will be parallel to l and can be extended to get l'.
Get students to think about why these steps of construction work.



In this example, the use of 'the construction for making an angle of equal measure' to construct the parallel lines can be justified using the theorem that "When two lines are intersected by a transversal in such a way that a pair of corresponding angles are congruent, the lines are parallel. The construction for copying an angle can be justified using the SSS Congruence Theorem.

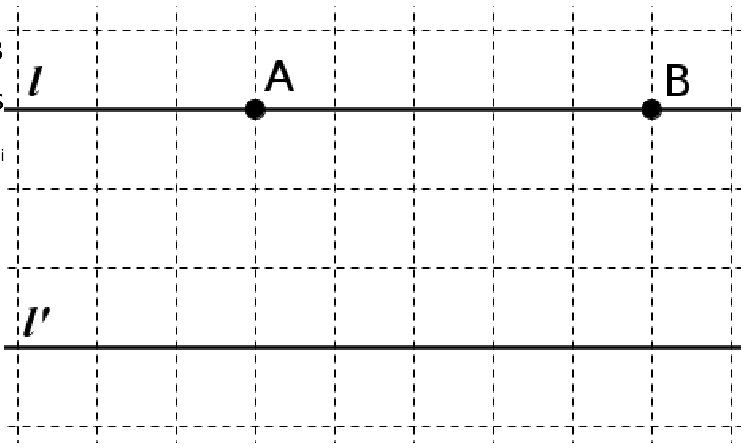
You may want to look at https://www.mathopenref.com/constparallel.html for the step by step construction and its justification.

There are alternate ways of doing the same constructions, using a different set of theorems. You can find two of them here.
https://www.mathopenref.com/constparallelrhombus.html

https://www.mathopenref.com/constparallelt t.html
Your students may come up with more alternatives! Interested students can go through these constructions as well and figure out the theorems used in those.
Also, notice that there are infinitely many lines through a point P. But the moment an additional condition – that it is parallel to l in this case, or that it is perpendicular to a given line, or that it passes through another point – is added the line gets uniquely determined. Thus two pieces of information are required to uniquely determine a line.

Task 2

Given parallel lines *l*, *l'* and points A and B on *l*: Construct at least 3 parallelograms ABC_1D_1, ABC_2D_2, ABC_3D_3, etc with points C_i and D_i on *l'* for i = 1, 2, 3, ...



Note: *All the points have to be grid points.*

Calculate the area of each of the parallelograms constructed by you.

Parallelogram	Base (unit) (AB)	Side (unit) (BC_i)	Area (unit^2)	Perimeter (units)
ABC_1D_1				
ABC_2D_2				
ABC_3D_3				

Compare the area of the parallelograms with those of the parallelograms drawn by your friends. What do you observe? What can say on the basis of your observations?

If we were to construct more parallelograms ABC_iD_i, what can say about their heights and areas? Would you need to measure its height or calculate its area? Discuss your reasons with your friends.

> Through this task, students are expected to realise that the area of parallelograms with the same base and between the same parallels are equal. That is given a base and height, the area remains a constant, irrespective of the length of the other side. Also help them see how the area of the parallelogram is related to the area of a rectangle with the same base and height. You can also ask them to calculate the area of the parallelograms to observe the relationship algebraically.

Task 3A

Now calculate the perimeter of each of the parallelograms that you drew and enter it in the appropriate column in the table above What is your observation about the perimeters of the different parallelograms drawn? What is the smallest possible perimeter?

> For a parallelogram with a given base and between two parallel lines, the shortest perimeter is obtained when the other side is the shortest distance between the given parallel lines. That is when the parallelogram is a rectangle. It would be good if students are encouraged to come to this conclusion through an exploration in GeoGebra. There may be other ways of visualising this relationship as well.

Task 3B

If you have a cabbage patch in the shape of a parallelogram of area 24 square metres. What are the possible base lengths and heights of the patch? How much fencing will be needed for each of these patches? When will the fencing material needed be minimum?

> In Task 3B, there could be many parallelograms with area 24 square units - (6,4), (2,12), (3,8), (16, $\frac{3}{2}$) are all possible (base, height) pairs. Students might see an obvious factorisation of 24 and conclude that only those are the correct answers. Encourage them to come up with multiple answers to the question. And realize that there are infinitely many (base, height) pairs for any given area.

For a given base and height, the minimum perimeter is obtained for a rectangle. Evaluating the perimeter of a few (base, sides pairs, height in case of rectangle) it is easy to see that the perimeter is smaller when the difference $|\text{base - height}|$ is smaller. Hence, the fencing will be minimum when the field is a square, with sides $2\times\sqrt{6}=2\sqrt{6}$ units.

Base - height pair	Minimum Perimeter possible
(6, 4)	20 units
($3(\sqrt{2})$, $4(\sqrt{2})$	$14(\sqrt{2})$ units
(8,3)	22 units
(12, 2)	28 units
(16, 3/2)	35

From such a table, students may come up with the insight that the perimeter is minimum when the base = height and the shape is a square. A GeoGebra exploration may be useful to visualise this.

Task 4

What is the minimum information needed to construct a unique parallelogram? (That is all of you get only congruent parallelograms)?

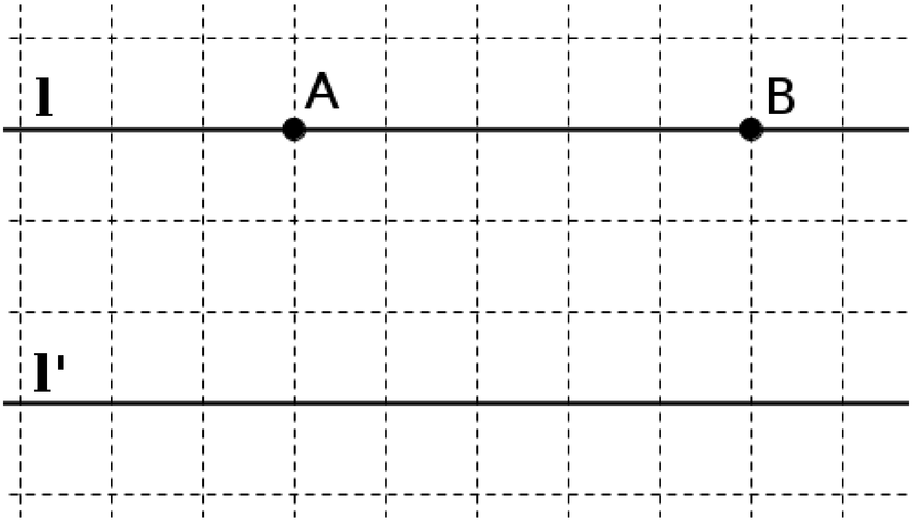
You can do this activity in pairs. Divide the students in teams of two and ask them how much and what information about a parallelogram would they need to give their friend to ensure that the parallelograms they both construct are congruent.

There are multiple answers to this question. One of the most common answer can be all four sides. But giving all four sides is not enough and one gets infinitely many parallelograms with all corresponding sides equal. If some students gives that answer ask them to construct a parallelograms with all four sides given and compare those with their friends.

If they say two adjacent angles and one side, try to ask them if they can get measures of all the angles of a parallelogram given one angle. Hence giving two angles and one side is the same as giving one angle and one side.

And hence length of the adjacent sides and one of the angles is necessary to construct a unique parallelogram. There are other sets of three independent conditions which will also give a unique parallelogram like 2 adjacent sides and the height or 1 side, 1 angle and the height.

Task 5



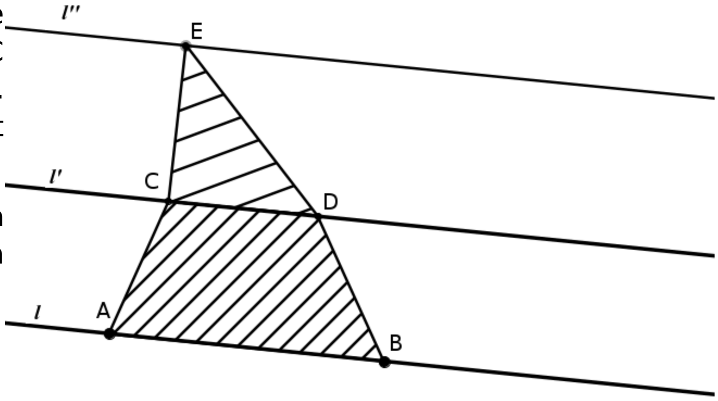
a) Draw three triangles with base AB, and the third vertex on *l*'. How do their areas compare? Justify your observation.

b) Draw three trapeziums with base AB and the opposite side on *l*'. Compare their areas. What can you say about the areas?

c) Can you draw two different trapezia such that their base is AB and the opposite side is on *l*' but the areas are the same? Justify your construction

Encourage students to note the difference when the length of sides of the trapezium on l' is a) varied and b) kept constant. Also help them relate this with the formula for area of a trapezium. Similarly sensitise them to the relation between the area formula for the triangle and the observation in task 5A.

Further Exploration on GeoGebra:



l, l’ , and l'' are parallel lines. A and B are two fixed points on l. Choose two points C and D on l’ to construct trapezium ABDC. What is the area of the trapezium that you constructed?

In GeoGebra, observe how the area changes as you move points C and D on line l’ in such a way that
a) length CD varies
b) length CD remains the same

Now choose a point E on l'' and construct triangle CDE. What is the area of the triangle? How do you think it would vary if the point E is moved on l''? Justify your answer.
What do you observe about the area of the composite figure ABDEC when the points C,D and E are moved in different ways?

References :

https://www.mathopenref.com/constparallelrhombus.html
https://www.mathopenref.com/constparallelt.html