# Mathematical Neighbours

**Overview:** This learning unit aims at motivating students about the famous 'Four Colour Theorem' through puzzles, a new way of representing maps and the technique of what is called 'Node Colouring.'

**Learning Objectives**

•To introduce mathematical modelling and creative problem solving

•To learn to represent a problem mathematically using graphs and the concepts of paths and colouring

**Materials required:**

Worksheets, Pens, Pencils, Erasers, Colour Boxes, Board, Projector

**Time required:** Three hours or wo sessions of 90 minutes.

**Recall:** Have you already completed the Finding the Right Path Learning Unit? It is a good idea to conduct the Finding the Right Path Learning Unit before conducting this unit.
Are you comfortable with basic definitions of nodes and links?

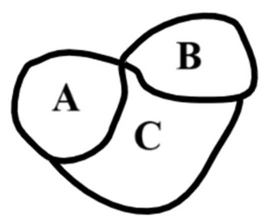
**Introduction:** Today we are going to solve some puzzles, draw some pictures, and colour some maps. While doing these activities, we will be learning a new and colourful way of representing real-life situations.

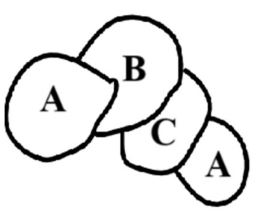
**Task 1A: One king and his kingdom!**

Once upon a time, there was a kingdom ruled by a king who had two daughters and one son. It was his wish that upon his death, this kingdom should be divided into three connected regions, one region for each child, such that all his three children are each other's neighbours. And the regions need not be equal.

*Note: Being neighbours means sharing some common border, which is more than just a single point. Also, the shares of each of the king's children need not be equal.*

In this map, region A and region B are not neighbours as they share only one point as a common border. Hence the king's wish is not satisfied.




In this map, region A is a neighbour of region B and region C but region A is not connected. Hence the king's wish is not satisfied.

**Q.** Can you draw another map such that the king's wish is satisfied? Compare solutions designed by your classmates.

**Task 1B: Second king and his kingdom!**

Now, there was another kingdom ruled by another king who had two daughters and two sons. He also had a similar last wish. It was his wish that upon his death, this kingdom should be divided into four connected regions, one region for each child, such that all his four children are each other's neighbours.

*Note: Being neighbours means sharing some common border, which is more than just a single point. Also, the shares of each of the king's children need not be equal.*

**Q.** Do you think that this is possible? Can you draw a map below such that the king's wish is satisfied? Draw a kingdom of any shape and look for a possible solution satisfying the King's wish. Is your solution similar to your classmates?

**Task 1C: Yet another king and his kingdom!**

There was a third kingdom ruled by another king who had three daughters and two sons. His wish was also similar to the earlier two kings. Like the other two kings, he also wished that upon his death, this kingdom should be divided into five connected regions, one region for each child, such that everyone is everyone's neighbour.

*Note: Being neighbours means sharing some common border, which is more than just a single point. Also, the shares of each of the king's children need not be equal.*

**Q.** Do you think that this is possible? Draw a kingdom of any shape and look for a possible solution satisfying the King's wish. Compare solutions designed by your classmates.

**Q.** Compare Tasks 1A, 1B and 1C. Could you draw maps for all the three? If not, which task were you unable to draw a map for? Discuss with your group members.

Discuss all the three tasks together. Give the students some time to think of different solutions. For every task, invite students to the board and show their solutions to their classmates. Give students time to think about the solutions.
In the case of Task 1C, students might get a feeling that they would not be able to divide a region in 5 parts and adjacent to each other. But they might not be able to articulate the reason.
Once they understand the task at hand, they themselves come up with the idea of dividing regions in different ways.

The only condition is that they have to share boundaries with each other. These figures illustrate some unsuccessful attempts to satisfy the third king's wishes. Let's take the first figure, region 1 shares a boundary with every other region. A similar thing can be noticed about region 2. But regions 3 and 5 do not share a boundary.
Some students might come up with similar figures given. But there are many ways, so please

don't stick to these figures only. After there is a consensus in the class that Task 1C doesn't have a solution, ask them to compare Tasks 1A, 1B and 1C and see what is changing.

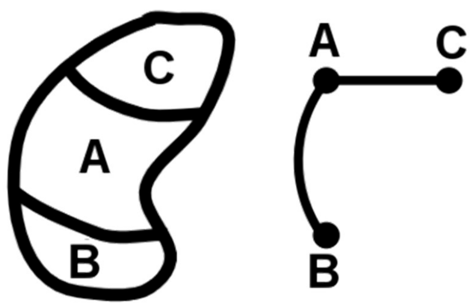
Let us now try to verify our conclusion through a new representation of the same problem.

### Task 2: Representing maps

A 'node' (drawn as a dot or point) represents an individual element (i.e., member) of the set of things in front of us. A 'link' (drawn as a connecting line) represents that, two nodes joined by that link are connected to each other. If there is no link connecting a pair of nodes, that means those two nodes are not linked to each other.

Let us call this new representation containing nodes and links a 'Network Diagram'.

Look at the regions and their Network Diagram given below. Let us say a kingdom is divided in three parts (A, B, and C) as shown in the diagram on the left. In the diagram on the right, each part of the kingdom is represented by a node (point) and the node pairs are connected by a link whenever the two regions share a common border.
In the diagram on the left, region A shares a border with regions B and C. Thus, in the diagram on the right, point A is linked to both B and C. But regions B and C do not share a border. Hence, in the Network Diagram, we can see that there is no link between nodes B and C.




Note that the links need not be straight lines, it can be curved as well like the one in the network diagram drawn above.

In Task 1, when you draw the Network Diagrams for the maps drawn by you, the regions of the kingdom will become the nodes and if two regions share a common border then those two nodes will have a link between them.

*Note that the link (connector) may not always be a straight line. Further, links are not allowed to intersect / meet each other at any points except the nodes.*

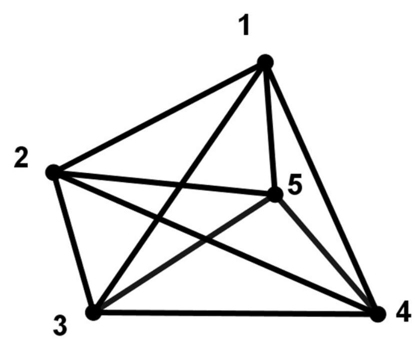
**Q.** Draw Network Diagrams for all the solutions you got for Task 1A. What can you say about the Network Diagrams of different solutions to the same problem? Are all the Network Diagrams of Task 1A the same?

**Q.** Draw Network Diagrams for all the solutions you got for Task 1B. What is your observation about the Network Diagrams of different solutions to the same problem?

First of all, ask them if these Network Diagrams are the same?
It may happen that they say no. Ask them why?
Give them hints like,
•Are the number of regions the same? (Which actually means are the number of nodes in the Network Diagram the same as the map?)
• Are their connections the same? (Which actually means if for any two regions which are neighbours, their corresponding nodes are connected with a link)
• There are chances that students say 'Yes'. Ask them their reasons behind their answers.
Although the solutions might look different, when you draw a corresponding Network Diagram for a solution, students will notice that the Network Diagrams are the same for all the solutions for the same problem. Except for the order of the labels, the figure will look the same.
If they still insist that the figures look different, hold a discussion to show that the nodes can be moved around to make one connection diagram look like another.

We have seen that for Task 1C, we cannot draw a map. The Network Diagram for Task 1C will look like the following figure if there is a solution.



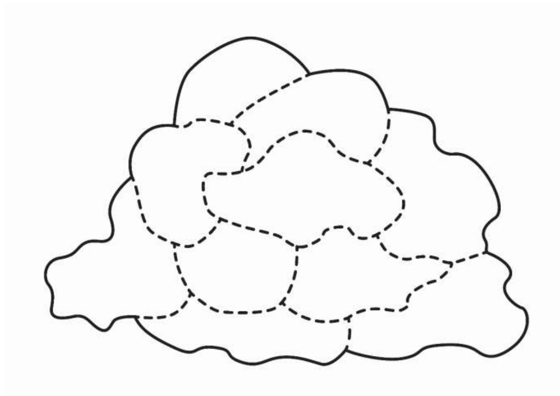

**Q.** Can you draw this Network Diagram for Task 1C without crossing the links (no links crossing each other)?

If one could divide the region in five parts such that each of them is a neighbour of all the others, then the Network Diagram would look like this. But in this diagram, one link is crossing other links, which we said earlier wasn't allowed. Through multiple trials, students will realise that it is impossible to draw a Network Diagram where all 5 nodes are joined to each other through links but no link crosses others.

**Task 3: Colouring Maps**
**Task 3A: Another country!**
See the map of an imaginary country below, where boundaries between different states are marked by dotted lines. Can you colour it in such a way that every state can be clearly distinguished from its neighbours?

Discuss with the students what can be their strategies to colour such that every state can be distinguished from its neighbour. At this point, you might have to give them a hint using the example of an Atlas where no neighbours have the same colour.

**Q**. Before you actually colour the map, estimate the minimum number of colours you will need. _______

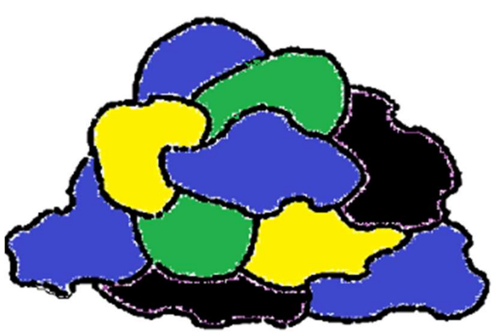


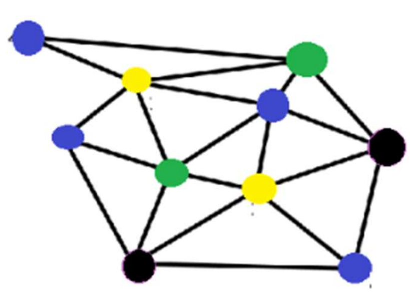
**Q**. How many colours did you actually need to colour the given map? ______
Compare your answer with the answers of your friends.

Make a table on board with two columns. First column will be the number of colours used (put numbers 1-10 in this column). In the second column you can put the number of students who concluded they needed those number of colours.
Note: We are not asking them what is the minimum number of colours anyone needed. We will just keep the table on the blackboard (don't erase) and revisit it during task 3B.

**Q.** Can you convert this map into a Network Diagram? Draw a Network Diagram representing this problem. Also colour the nodes of the Network Diagram. Remember that if a pair of nodes is connected by a link, that means they are neighbours on the map. So, they cannot have the same colour.

This is one solution for the given map. There can be many more solutions or representations that the students might get.





First of all, ask why these representations are the same?
It may happen that they say no. Ask them why?

Give them hints like,
• Are the number of nodes the same?
• Are their links the same?
• Also, is the number of colours used to code the Network Diagram the same as the number of colours used to colour the map?
• There are chances that students say 'Yes'. Ask them their reasons behind their answers.

**Task 3B: Colours of India**

**Q.** Have you ever seen a coloured map of India? What have you noticed in it?
Students might respond with 'states', 'districts', 'sea' and so on.

If you notice any map in the textbooks, atlas or on the internet, neighbouring regions (say states in a map of India) are always coloured in different colours.

**Colouring the neighbourhood of a state.**

Now look at the state of Manipur. From the map we can see that Manipur has three neighbours, namely Nagaland, Assam and Mizoram. We can also see that Assam and Nagaland are neighbours and so are Assam and Mizoram but Nagaland and Mizoram are not neighbours. How many colours do we need to colour the neighbourhood map of Manipur if we have to strictly follow the rule that any two neighbours must have different colours?

Remember how you coloured the imaginary kingdom in Task 3A? You can use the same strategies here too.

**Q.** What is the minimum number of colours you need to colour the neighbourhood map of Manipur? ____________________

**Now choose another state of your choice and colour the neighbourhood of that state.**

**Q**. How many colours will you need to colour the neighbouring states of the state you chose, such that any two states which are neighbours of each other have different colours or such that no two 'neighbouring' states have the same colours?

**Task 4**: **Network Diagrams and gathering of people**

In this task, we will see examples how the Network Diagram can help in modelling and solving various situations.

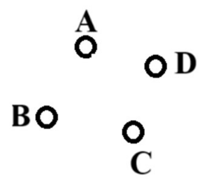
Imagine that you are at a gathering. And the organiser of this party wants it to be as colourful as possible. Thus, she gives the following instruction to all her guests: 'Talk to all the other guests you know and choose a dress of a colour different from their dresses'. As a result, we will have a situation where any two persons who know each other would have clothes of different colour.

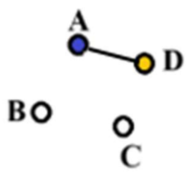
 

1. **Draw Network Diagrams for all the following gathering situations and find out how many different coloured clothes would be needed in each of the parties**.

   **A.** If there are 4 people (A, B, C, and D) in the party. A and D know each other and nobody else anybody. What is the minimum number of different coloured clothes that you expect to see at this party??

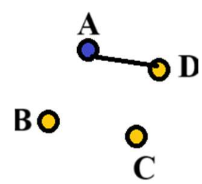
   The gathering has 4 people.




A and D know each other so we draw a link between them and they will wear different coloured clothes.

The others can wear either of the two colours.




A and D know each other so they have to wear clothes of different colours. But B and C do not know anybody so they can wear clothes whose colour is the same as A or D. Hence only two colours are enough.

   **B.** If there are 4 people (A, B, C, and D) in the party and all people in the party know each other, how many different coloured clothes would be needed?

Four colours. Everybody has to wear a different coloured dress.

   **C.** If there are 5 people (A, B, C, D and E) in the party. A, B and C know each other and D and E know each other. How many different coloured clothes would be needed?

Three colours

   **D.** If there are 5 people (A, B, C, D and E) in the party. A and B know each other, B and C know each other, D and E know each other and C and E know each other. How many different coloured clothes would be needed?

Two colours

   **E.** If there are 5 people (A, B, C, D and E) in the party and all people in the party know each other, how many different coloured clothes would be needed?

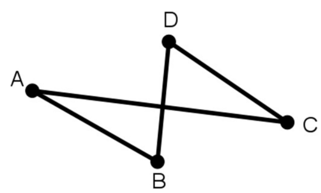
Five colours

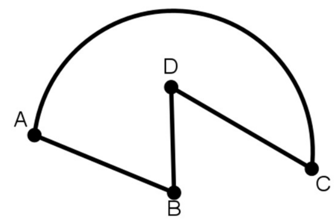
**Task 5: Relation between nodes, links and colours**

Let us see if there is any relation between the crossing and non-crossing links of a Network Diagram and minimum number of colours needed for that Network Diagram such that ‘neighbouring’ nodes have different colours.

**Q.** Look at the Network Diagram given here. You will notice that some of the links are crossing each other. Are the two Network Diagrams given below the same, i.e., do they represent the same situation?



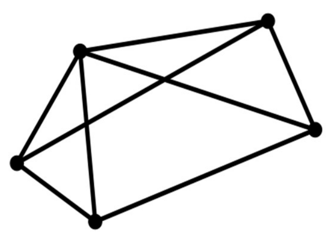

**Links are crossed**



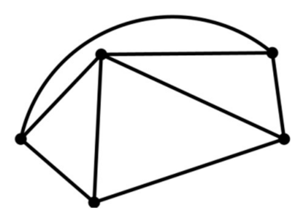

**Links are not crossed**

You can ask the students if the above two Network Diagrams are the same or not. Ask them to give reasons for their answers. The Network Diagrams are the same. This can be checked by looking at the number of nodes, links and the connections

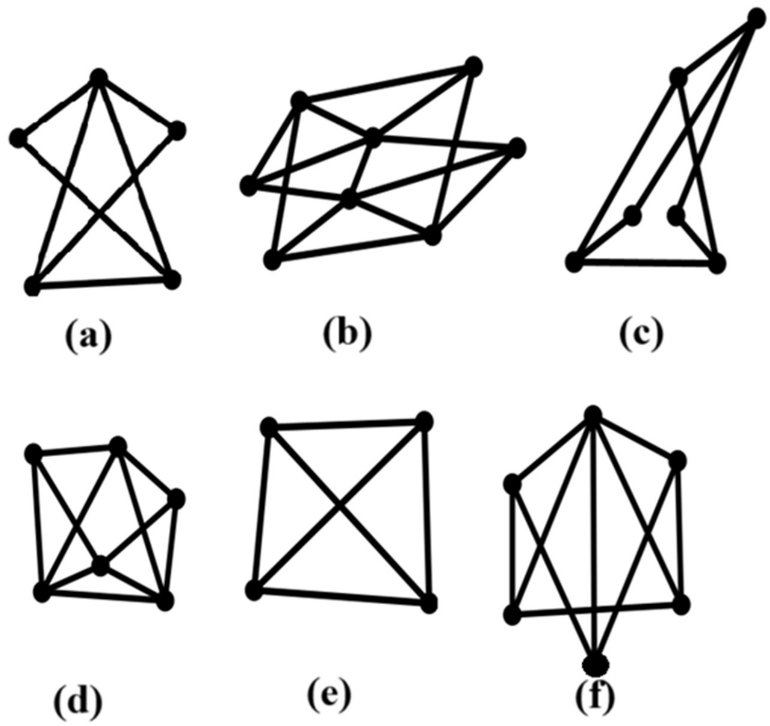
**Q.** Can you try to draw this Network Diagram such that no links cross each other?



The above Network Diagram can be drawn like this to ensure that the links do not cross.



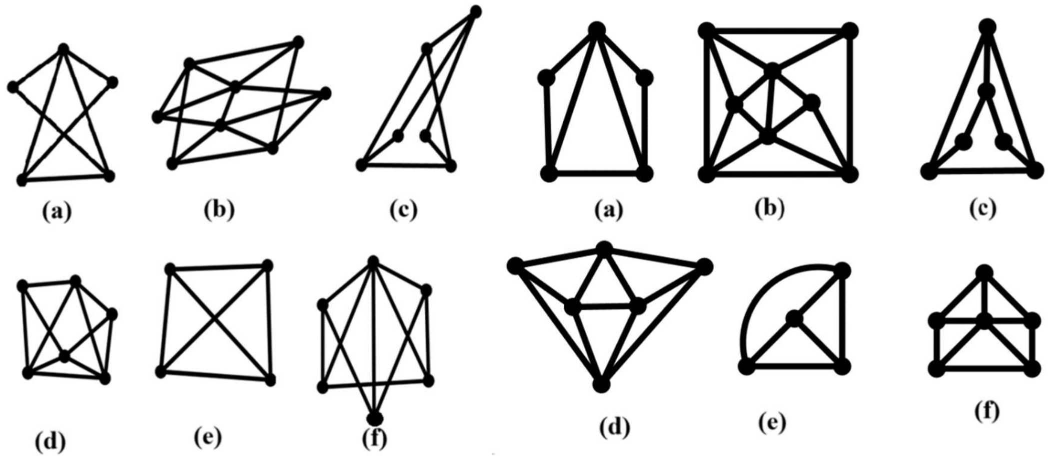
**Set 5A:** See the set of Network Diagrams given below. Fill the table.
(*'Neighbouring' nodes are nodes which have a link between them*)




| Network Diagram | Total number of nodes | Minimum number of colours needed to colour the nodes such that 'neighbouring' nodes have different colours | Can you draw it without the links crossing each other? |
|---|---|---|---|
| (a) | | | |
| (b) | | | |
| (c) | | | |
| (d) | | | |
| (e) | | | |
| (f) | | | |

**Q.** Did you notice any pattern in the table?

All the Network Diagrams given above can be coloured using four colours or less and all of them can be drawn without links crossing each other.



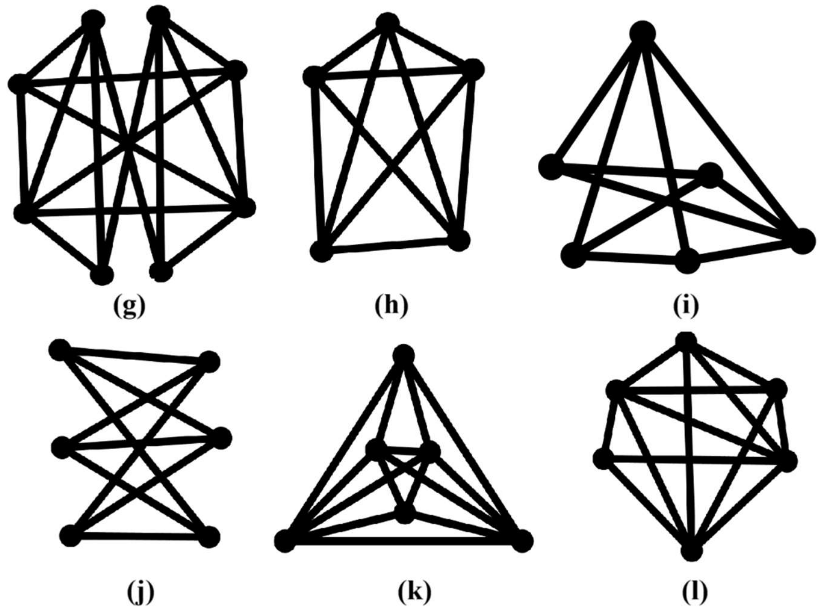

| Network Diagram | Total number of nodes | Minimum number of colours needed to colour the nodes such that 'neighbouring' nodes have different colours | Can you draw it without the links crossing each other? |
|---|---|---|---|
| (a) | 5 | 3 | Yes |
| (b) | 8 | 3 | Yes |
| (c) | 6 | 3 | Yes |
| (d) | 6 | 3 | Yes |
| (e) | 4 | 4 | Yes |
| (f) | 6 | 4 | Yes |

All the Network Diagrams given above can be drawn without their links crossing each other. Such Network Diagrams are examples of what is known as 'Planar Network Diagrams'. Maps of regions drawn on paper can be always represented by planar Network Diagrams. Let us see if all Network Diagrams can be drawn such that their links do not cross.

**Q.** Can we find some Network Diagrams where it is impossible to draw it without the links crossing each other?

**Set 5B:** Fill the table for these Network Diagrams.
(*'Neighbouring' nodes are nodes which have a link between them*)




| **Network Diagram** | **Total number of nodes** | **Minimum number of colours needed to colour the nodes such that 'neighbouring' nodes have different colours** | **Can you draw it without the links crossing each other?** |
|---|---|---|---|
| (g) | | | |
| (h) | | | |
| (i) | | | |
| (j) | | | |
| (k) | | | |
| (l) | | | |

**Q.** Did you notice any pattern in the table?

| Network Diagram | Total number of nodes | Minimum number of colours needed to colour the nodes such that 'neighbouring' nodes have different colours | Can you draw it without the links crossing each other? |
|---|---|---|---|
| (g) | 8 | 4 | No |
| (h) | 5 | 5 | No |
| (i) | 6 | 3 | Yes |
| (j) | 6 | 2 | No |
| (k) | 6 | 5 | No |
| (l) | 6 | 5 | No |

In the set 6B, we saw some Network Diagrams where drawing them without links crossing is impossible. Such Network Diagrams are called non-planar Network Diagrams as opposed to planar Network Diagrams we saw earlier. When you draw non-planar Network Diagrams on a 2-D surface (i.e., paper), the links will necessarily intersect each other and you cannot avoid the intersections of links by moving the nodes.

**Q.** Did you see any relation between Network Diagrams which can be drawn without the links crossing and the number of colours that are necessary to colour the Network Diagram?

In some non-planar Network Diagrams, you do require more than 4 colours. Your students might notice some relation between Network Diagrams that can be drawn without their links crossing and the number of colours needed to colour the Network Diagram. Ask them to draw more Network Diagrams and check if the patterns hold for them too.

At the end of this LU, the students may have realised that the planar Network Diagrams (the ones that can be drawn without their links crossing) can be coloured using four or less colours. And we mentioned earlier that any Network Diagram corresponding to a physical map is a planar network diagram. This is the famous 'Four Colour Theorem' which states that "Every planar Network Diagram can be coloured using four or less colours such that no adjacent nodes have the same colour". It was first proposed in 1852 by Francis Guthrie, a mathematician, while coloring a map of England. For over 100 years, mathematicians struggled to prove it. In 1976, Kenneth Appel and Wolfgang Haken finally showed it was true, using a combination of math and computers. This was one of the first major proofs where computers played a big role, making it a significant moment in mathematical history!