Mid-Point Quadrilaterals

Overview:

In this unit, students will use GeoGebra, an open-source dynamic geometry software to explore and make conjectures about the properties of the quadrilateral obtained by joining the mid-points of another quadrilateral. They will also look at some special cases of the mid-point quadrilateral theorem and prove the special cases.

Minimum time required:

Minimum 2 sessions of 40 minutes (After the students have practiced GeoGebra)

Type of Learning Unit:

Computer Laboratory

Links to the curriculum

NCERT Class 8	Chapter 3 – Understanding Quadrilaterals
NCERT Class 9	Chapter 8 – Quadrilaterals

Learning Objectives:

Exploring the properties of quadrilaterals using GeoGebra

Conjecturing, Verifying, and Proving the mid-point quadrilateral theorem

Making conjectures about the special cases of the mid-point quadrilateral theorem and proving them

Prerequisites:

Familiarity with the basic construction tools in GeoGebra

Properties of special quadrilaterals

Materials Required:

Computers (students may work in pairs) with GeoGebra Classic 5 installed, Papers, Pencils/Pens

Suggested Readings

- https://free.openeclass.org/modules/document/file.php/SC245/GeoGebra-in10Lessons.pdf
- https://www.youtube.com/watch?v=1cBXWi66-tY (*A Geogebra Manual*) (*A GeoGebra Tutorial*)
- https://demonstrations.wolfram.com/TheMidpointQuadrilateralTheorem/
- https://mathforlove.com/2012/02/midpoints-of-a-quadrilateral-form-a-parallelogram/
- https://www.techhouse.org/~mdp/midpoint/nonquad.php

Introduction:

In this learning unit, you are going to use a software app (or application) called GeoGebra to explore the quadrilaterals formed by joining the midpoints of any given quadrilateral. You are also going to make many conjectures about special cases of these quadrilaterals using GeoGebra and are going to try and prove the conjectures.

Sketch and Investigate

Open a new page in GeoGebra. Click on View and then on Axes to hide the axes. Only the algebra and graphics views should be visible.

Task 1

Construct a quadrilateral ABCD using the Polygon Tool.

To do this click on the **Polygon tool** and click the points in the following order: point *A*, point *B*, point *C*, point *D*, and then point *A* to close the polygon.

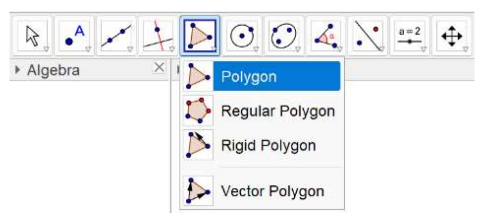

Figure 1

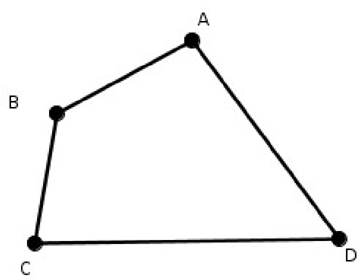

Figure 2

Did you notice that GeoGebra labels the vertices with uppercase letters and the line segments with small case letters? This is a common convention.

Right-click on each of the sides of the quadrilateral ABCD and unselect Show Label from the menu to hide the labels a, b, c, and d of the sides.

To avoid cluttering the figure with too many labels, go to Options in the main menu, and select 'Labeling' followed by 'New Points Only'. This will ensure that new line segments are not labeled.

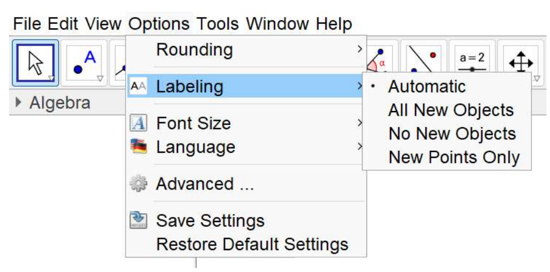

Figure 3

Find the midpoints of the sides, AB, BC, CD, and AD. Some of them are marked here. This can be done by selecting the Midpoint or Center icon from the Point Tool menu and then clicking on the four sides of the quadrilateral ABCD.

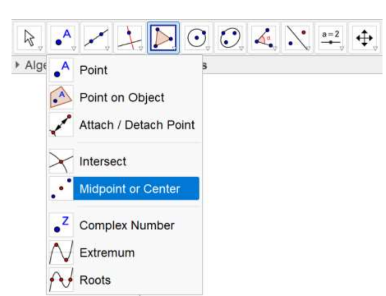

Figure 4

The midpoints will be labeled as E, F G, and H respectively.

Make the new quadrilateral by selecting the Polygon tool and then clicking on the points E, F, G, and H in that order. This will be called the midpoint quadrilateral. By a mid-point quadrilateral, we mean a quadrilateral formed by joining all the midpoints of the sides of a given quadrilateral.
Drag the vertices of the original quadrilateral ABCD and observe what happens to the midpoint quadrilateral EFGH. Record your observations below.

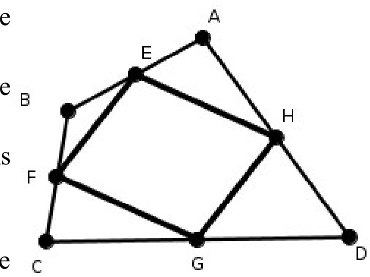

Figure 5

Is there any similarity between all the mid-point quadrilaterals you got while dragging the vertices?

If the students are unable to notice that e = g and f = h bring it to their attention but before doing so give them enough time.

Note the lengths of the four sides of quadrilateral EFGH (which are named as e, f, g, and h in the Algebra view). What do you observe? Now drag any one of the vertices of the original quadrilateral ABCD. What are your observations regarding e, f, g, and h? Based on the observations, what can you say about them?

What kind of quadrilateral do you think EFGH is? Use your observations to support your conjecture.

In the above task, the students are asked to look at lengths of opposite sides of the mid-point quadrilateral, another way to check that the quadrilateral EFGH is a parallelogram is by checking the angles of the quadrilateral using the angle tool.

You can ask the students to drag the vertices to check if both pairs of opposite angles are equal or not.

You can also ask the students, for which quadrilaterals this happens.
The students can observe different aspects of the mid-point quadrilateral which points out that it is a parallelogram. Some of them are

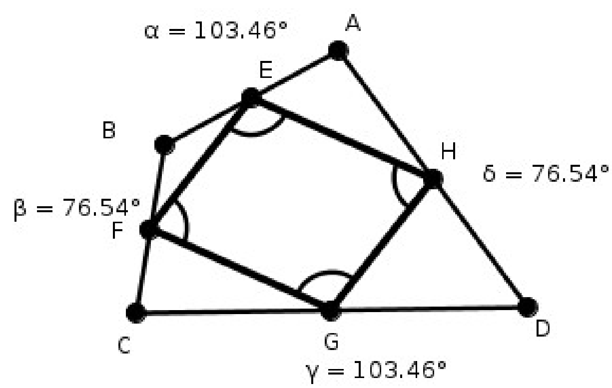

Figure T1

1) Both pairs of opposite sides are equal
2) Both pairs of opposite angles are equal
3) The diagonals bisect each other

All of these characteristics can be verified using GeoGebra.
The diagonal AC divides the quadrilateral ABCD into two triangles ΔABC and ΔACD. In each of these triangles, one of the sides of the midpoint quadrilateral is a mid-segment (segment joining the midpoints of the other two sides).
And use this information to validate the conjecture made by you.

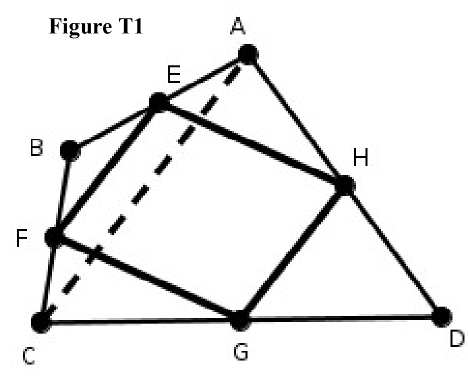

Figure T2

At this stage, some students might say that EFGH is a trapezium. Use this opportunity to discuss the definition of trapezium and ask them what will happen if they look at diagonal BD.

Would you be able to make similar conjectures by considering the diagonal BD of quadrilateral ABCD instead of diagonal AC?

Task 2

Prove your conjecture.

Here EFGH is the midpoint quadrilateral of the quadrilateral ABCD.

AC and BD are diagonals of the quadrilateral ABCD.

Proof of EFGH is a parallelogram:
EF || AC, also GH || AC. So, EF || GH
Similarly,
EH || BD, also FG || BD. So, EH || FG
Hence EFGH is a parallelogram.

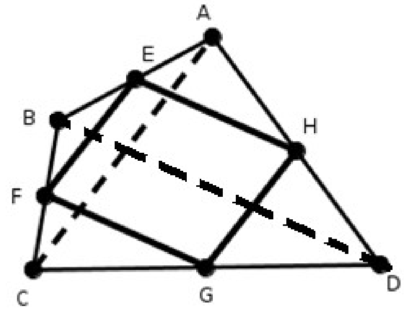

Figure T3

Task 3

Observe the numbers associated with Poly1 (ABCD) and Poly2 (EFGH) in the Algebra view. Can you see a relationship between them?

From this, what can you say about the area of the midpoint quadrilateral EFGH in relation to the area of ABCD?
You can ask the children to observe the Poly 1(ABCD) and Poly 2 (EFGH) in the Algebra view and ask them whether they see a relationship between these two.

Draw the midpoint quadrilateral of EFGH. GeoGebra will label it as IJKL. What kind of a quadrilateral is IJKL? How is its area related to that of EFGH and ABCD?

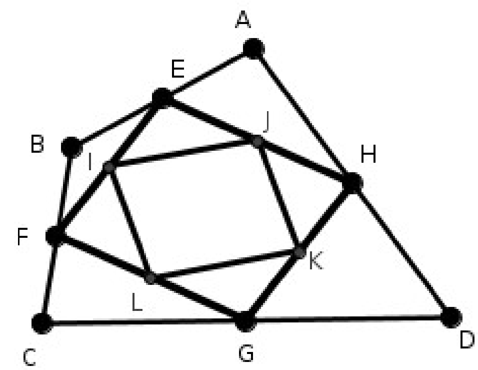

Figure 6

Continue drawing midpoint quadrilaterals as you did in Task 1.
Can you see how these midpoint quadrilaterals are related to each other?

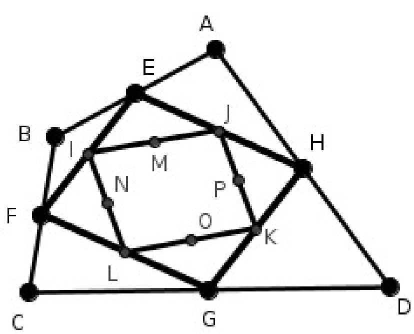

Figure 7

Task 4

Prove your conjecture.

Here EFGH is the midpoint quadrilateral of the quadrilateral ABCD.
AC and BD are diagonals of the quadrilateral ABCD.

Proof of Area of EFGH = $\frac{1}{2} \times$ (Area of ABCD):

Now, look at triangles ABC and ACD

Area of (Triangle BFE) = $\frac{1}{4} \times$ Area (Triangle ABC) … (Midpoint triangle) ---- (1)

Area of (Triangle DGH) = $\frac{1}{4} \times$ Area (Triangle ACD) … (Midpoint triangle) ---- (2)

Similarly, in triangles BCD and ABD

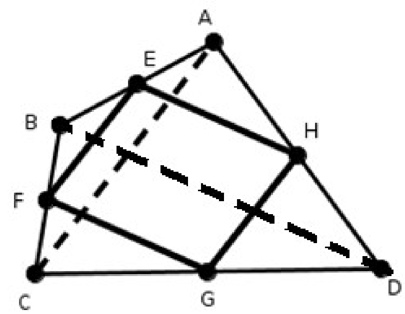

Figure T4

Area of (Triangle AEH) = $\frac{1}{4} \times$ Area (Triangle ABD) … (Midpoint triangle) ---- (3)

Area of (Triangle CFG) = $\frac{1}{4} \times$ Area (Triangle BCD) … (Midpoint triangle) ---- (4)

So, Area EFGH = Area of ABCD – (Area of BEF) – (Area of AEH) – (Area of DGH) – (Area of CFG)

Area EFGH

= Area of ABCD – $\frac{1}{4} \times$ [(Area ABC) + (Area ACD) + (Area of ABD) + (Area of BCD)]

= Area of ABCD – $\frac{1}{4} \times$ [{ (Area ABC) + (Area of ACD) } + { (Area ABD) + (Area of BCD) }]

= Area of ABCD – $\frac{1}{4} \times$ [(Area ABCD) + (Area ABCD)]

= Area of ABCD – $\frac{1}{2} \times$ (Area ABCD)

= $\frac{1}{2} \times$ (Area ABCD)

For the tasks given below ask the students to discuss the ways they would construct a rectangle or a rhombus or a square using GeoGebra. You can help them by asking them to recall various properties they know of these quadrilaterals. All the mid-point quadrilaterals (parallelograms) that the students will get in the constructions below will be a special kind. Inside a rectangle, they will get a rhombus and inside a rhombus, they will get a rectangle. Inside the square, they will get a square. Encourage them to make these conjectures and prove them. Also, highlight the fact that by using GeoGebra they are only formulating conjectures and verifying them and not proving them.

Task 5

Using GeoGebra, draw a rectangle. Move the vertices of the rectangle you have drawn and check if the rectangle remains a rectangle. If not, draw again.

Write down the steps you took to ensure that you have drawn a rectangle.

Draw a mid-point quadrilateral of this rectangle. What can you say about this mid-point quadrilateral? Can you prove your conjecture?

Constructing a rectangle using GeoGebra: We need to construct such a rectangle which will remain a rectangle even when you move its points. For this we will need to use construct this rectangle using the various properties of rectangles we know; like all pairs of adjacent sides are perpendicular to each other or all angles are equal.

For this, you would need to use the perpendicular line tool. Draw a line segment AB on the geometry pad and then draw two perpendicular lines on its two ends. Then select a point on the perpendicular line from B using the Point Tool, which will be Point C. And then draw another perpendicular line from C to the perpendicular line from Point B. Select the point where this line intersects the perpendicular line from A. This quadrilateral ABCD is a rectangle. Move A, B, or C and check if this remains a rectangle after moving.
Can you find other ways of constructing a rectangle?

(Hint: *The diagonals of a rectangle are always equal and intersect at their mid-points.*)

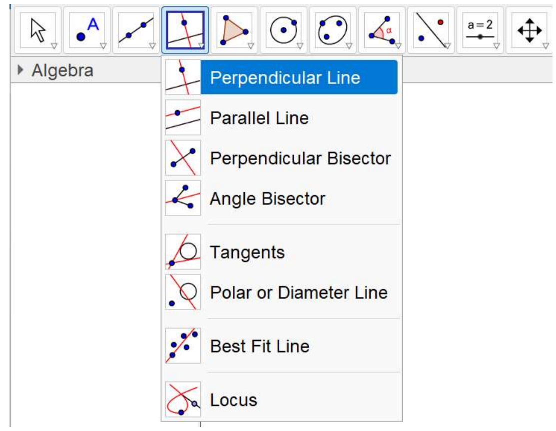

Figure 8

Proof of the statement:
"**Midpoint Quadrilateral of a rectangle is a rhombus**"
Given that ABCD is a rectangle. And E, F, G, and H are the midpoints of the four sides, AD, AB, BC, and DC respectively.
So, AE = ED = BG = CG and AF = BF = CH = DH.
Also, the four triangles (AEF, DEH, CGH, and BFG) are all right-angle triangles.
Hence, EF = EH = HG = FG (Pythagoras Theorem and SAS Property)
Hence, we have that EFGH is a rhombus.

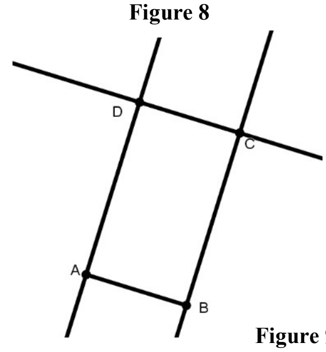

Figure 9

Task 6

Using GeoGebra, draw a rhombus. Move the vertices of the rhombus you have drawn and check if the rhombus remains a rhombus. If not draw again.

Write down the steps you took to ensure that you have drawn a rhombus.

Draw a mid-point quadrilateral of this rhombus. What can you say about this mid-point quadrilateral? Can you prove your conjecture?

Constructing a rhombus in GeoGebra: To be able to construct a rhombus that will remain a rhombus even after you move any of its points, you would have to construct a quadrilateral using the properties of a rhombus, that means using the fact that all sides are equal or the diagonals of a rhombus bisect each other at right angles. Here we will use the second property.

For this, you would need to use the perpendicular bisector tool.

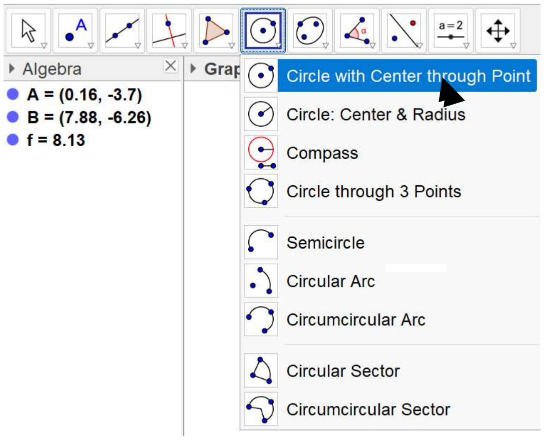

Figure T6

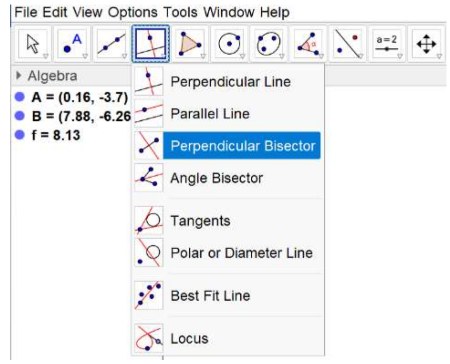

Figure T5

Draw a line segment AB on the geometry pad and then draw its perpendicular bisector using the perpendicular bisector tool. Then use the circle tool to draw a circle using the intersection point of the AB and its perpendicular bisector as a centre and a point of the perpendicular bisector, call it C. Then select the other point of intersection of this circle and the perpendicular bisector, call it D. Construct the polygon with A, C, B, and D. This quadrilateral ACBD is a rhombus. Move A, B, or C and check if this remains a rhombus after moving.

Can you find other ways of constructing a rhombus?

Proof of the statement:
"**Midpoint Quadrilateral of a rhombus is a rectangle**"

Given that ABCD is a rhombus. And E, F, G, and H are the midpoints of the four sides, AB, BC, CD, and AD respectively.
Now, the diagonals of a rhombus are perpendicular bisectors of each other. So, $AC \perp BD$.
Also, $EF \parallel AC \parallel GH$ and $EH \parallel FG$
---- Midpoint theorem for triangles
So, $EF \perp EH$ and $GH \perp FG$
Hence, we get, $\angle FEH = \angle EFG = \angle FGH = \angle GHA = 90^o$
So, we have that EFGH is a rectangle.

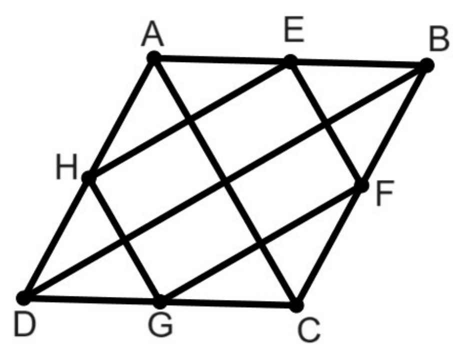

Figure T7

Task 7

Using GeoGebra, draw a square. Move the vertices of the square you have drawn and check if the square remains a square. If not draw again.

Write down the steps you took to ensure that you have drawn a square.

Draw a mid-point quadrilateral of this square. What can you say about this mid-point quadrilateral? Can you prove your conjecture?

A square is always a rectangle (All angles are equal).
So, the midpoint quadrilateral of a square is a rhombus. Hence all its sides are equal.
Similarly, a square is always a rhombus (All sides are equal).
So, the midpoint quadrilateral of a square is a rectangle. Hence all its angles are equal.
A quadrilateral whose all sides and all angles are equal is a square.
So, the midpoint quadrilateral of a square is a square.

References:

Lingefjärd, T., Ghosh, J, Kanhere, A. (2015). Students Solving Investigatory Problems with GeoGebra - A Study of Students' Work in India and Sweden. In S.J. Cho (Ed.),
The Proceedings of the 12th International Congress on Mathematical Education

