Counting Areas

Overview

In this unit, students are led through a guided discovery of the Pick's Theorem and its proof. It invites them to consider the relationship between the areas of special cases of grid polygons[1] and see if it can be generalised to any grid polygon or how it should be modified to be applicable to the general case. Proofs for special cases are considered and directions are given for a general case.

Learning Objectives

To provide opportunities for students to engage in practices of mathematics such as

make conjectures,
modify or refine them based on additional information,
generate examples to verify
refute conjectures through counter examples,
prove conjectures,
consider special cases,
generalise them etc.

Material Required

Worksheet/Grid Paper

Time Required

3 sessions of 40 minutes each

Prerequisites

Students should be familiar with basic understanding of area of polygons.

Introduction: In this unit we will be finding a new way to get areas of polygons. Here we will not be calculating or approximating areas of polygons but figuring newer ways to find areas of polygons.

Let us start with a story.

King Bahubali and his elephants

King Bahubali loved elephants so much that he kept a herd of them. In fact, he planted his coconut garden in such a way that it looked like an elephant when viewed from his terrace!
But the elephants would walk around the garden and destroy it. So, the king put a fence around the garden to keep the elephants away as seen in Figure 1. The trees were planted on a square grid, with one tree at each grid point; the distance between any two consecutive grid points was 1 unit, to provide sufficient space for each tree. Can you find the area of the coconut garden?

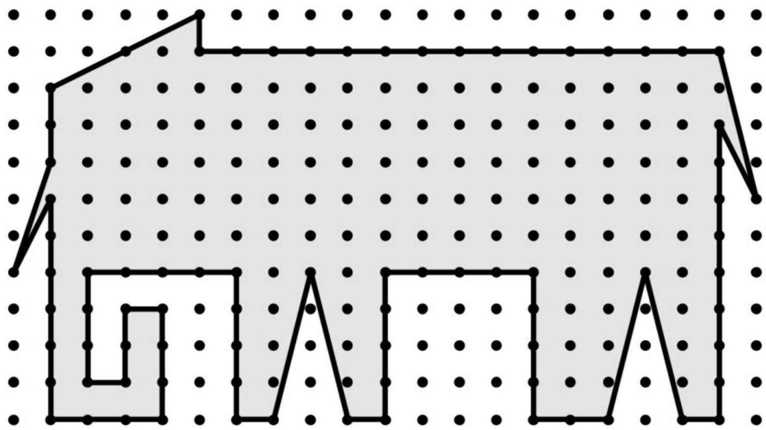


Figure 1

If you cannot solve it now, go ahead with the remaining tasks, and you will be able to do this at the end of the tasks!!

[1] Grid polygons: Polygons whose vertices are points on a square grid

This task is meant as a motivation for the unit. Students are not expected to solve the problem at this point. They can come to the solution of this later, after engaging with the first few tasks of the learning unit.

Task 1:

Given below are some shapes. Find the area of each and complete the given table.

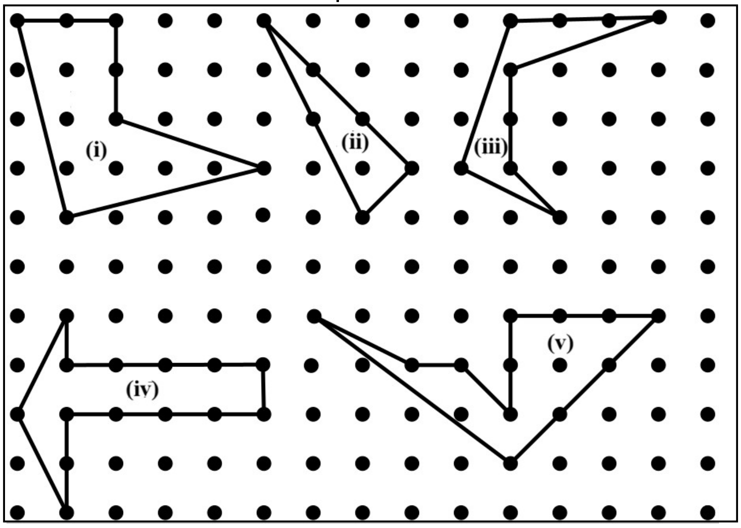
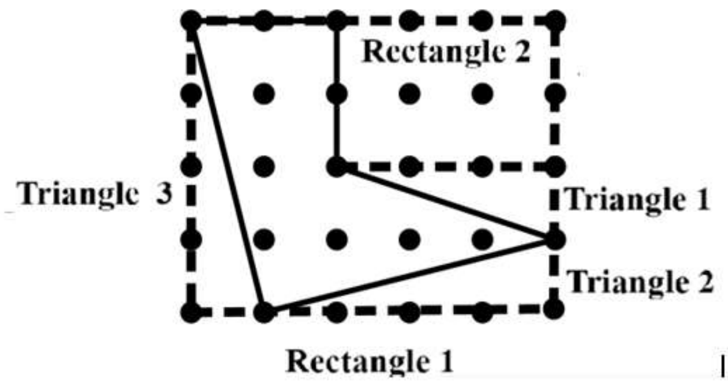


Figure 2

Shape	Area in Sq Units
(i)	
(ii)	
(iii)	
(iv)	
(v)	

Table 1

Observe how students are finding the area of the shapes. Encourage them to use multiple ways of finding areas – say counting grid squares, breaking up the shape into different shapes whose areas can be easily found out, or looking for 'part-squares' which add up to a complete grid-square etc. For example:



Area (Shape 1) = Area (Rectangle 1) – [Area (Triangle 1) + Area (Triangle 2) + Area (Triangle 3) + Area (Rectangle 2)

Task 2: Some more shapes!

a) Find the area of the following shapes. Also count the number of grid-points on the boundary of each shape in the Figure 3, and fill Table 2.

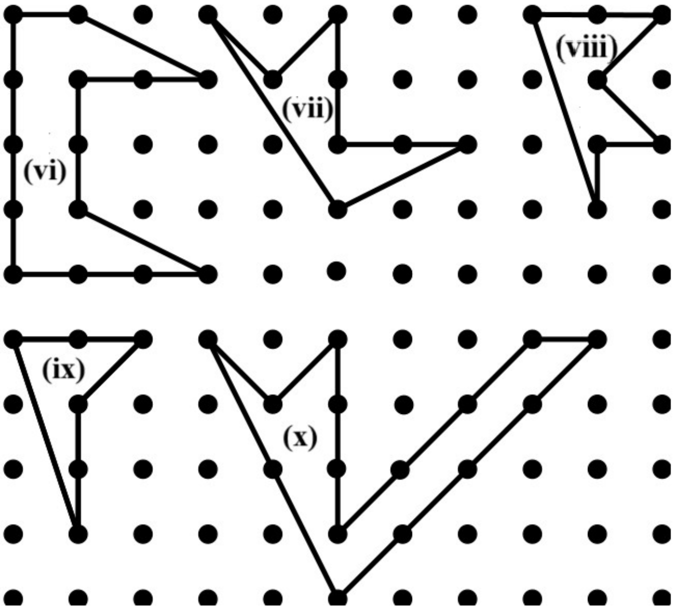


Figure 3

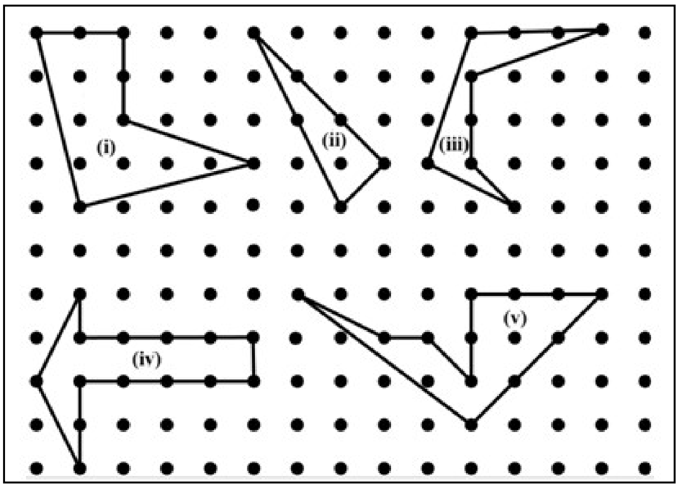
Shapes	Area in Sq Units (A)	Number of grid-points on the boundary (B)
(vi)		
(vii)		
(viii)		
(ix)		
(x)		

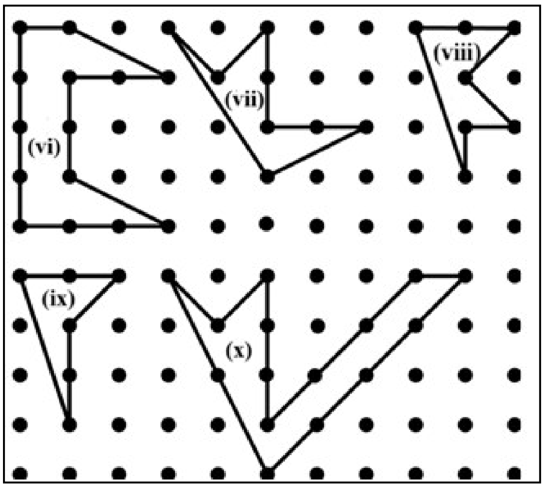
Table 2

b) Do you see any relation between the area of a shape and the number of grid-points on its boundary?

c) Does the same relation hold for shapes (i) to (v) in Task 1? If not, for which ones does the relation hold?

Let us look at shapes from Task 1 and Task 2 together and compare.





Shape	Area in Sq Units	Number of grid-points on the boundary (BP)	

(i)			
(ii)			
(iii)			
(iv)			
(v)			

(A) The relation holds for Shapes numbers ________________.

(B) The relation does not hold for Shapes numbers ________________.

The area of the shapes = $\frac{B}{2}$ - 1, where B is the number of grid-points on the boundary. The relation holds for those shapes that do not have interior grid-points. The relation holds for shapes (iii) and (iv) of Task 1.
The table in Task 1 could be extended and appropriate column can be added to find this out.

Task 4: Finding the expression!

a) In Task 3c, how do the shapes in (A) and (B) differ?

b) How would you modify the relation in Task 3b such that it holds for all shapes?

In Task 3a the distinguishing characteristic of shapes (iii) and (iv) is these shapes do not have grid-points inside them, whereas the other shapes have.

Observe what properties students come up with and if it is verifiable that all shapes that have this property satisfies the relation, $\text{Area} = \frac{B}{2} - 1$. If not provide, counterexamples of shapes which do not satisfy the relation. This can be done by other students as well.

Provide sufficient time for students to engage with task 3b. In case they are not able to come up with a modified relation, you may want to provide hints such as the following.

1) In the following figure, Triangles A and B, both of which have no grid-points in their interior can be put together to make triangle C, which has one grid-point in the interior.

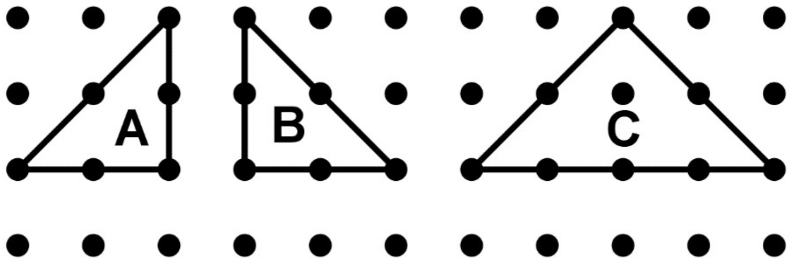


Figure T1

Which points on the boundary of triangles A and B are also on the boundary of C? Do some points on the boundary become interior grid-points now? Do some grid-points coincide when the triangles are put together?

2) Let students tabulate the number of grid-points in the interior, on the boundary and the difference between the area and the expression, $\frac{B}{2} - 1$. It can be seen that the correct formula is $\text{Area} = I + \frac{B}{2} - 1$, where I is the number of grid-points in the interior.

Task 5: Making some more shapes

Draw five more shapes on the grid provided below and check if the relation holds for these shapes as well.

(a) Are you sure that it will hold for any shape that you may draw on the grid?
(b) What are the properties common to the shapes for which this relation holds?
(c) Did your relation hold for all the shapes you drew on the grid?

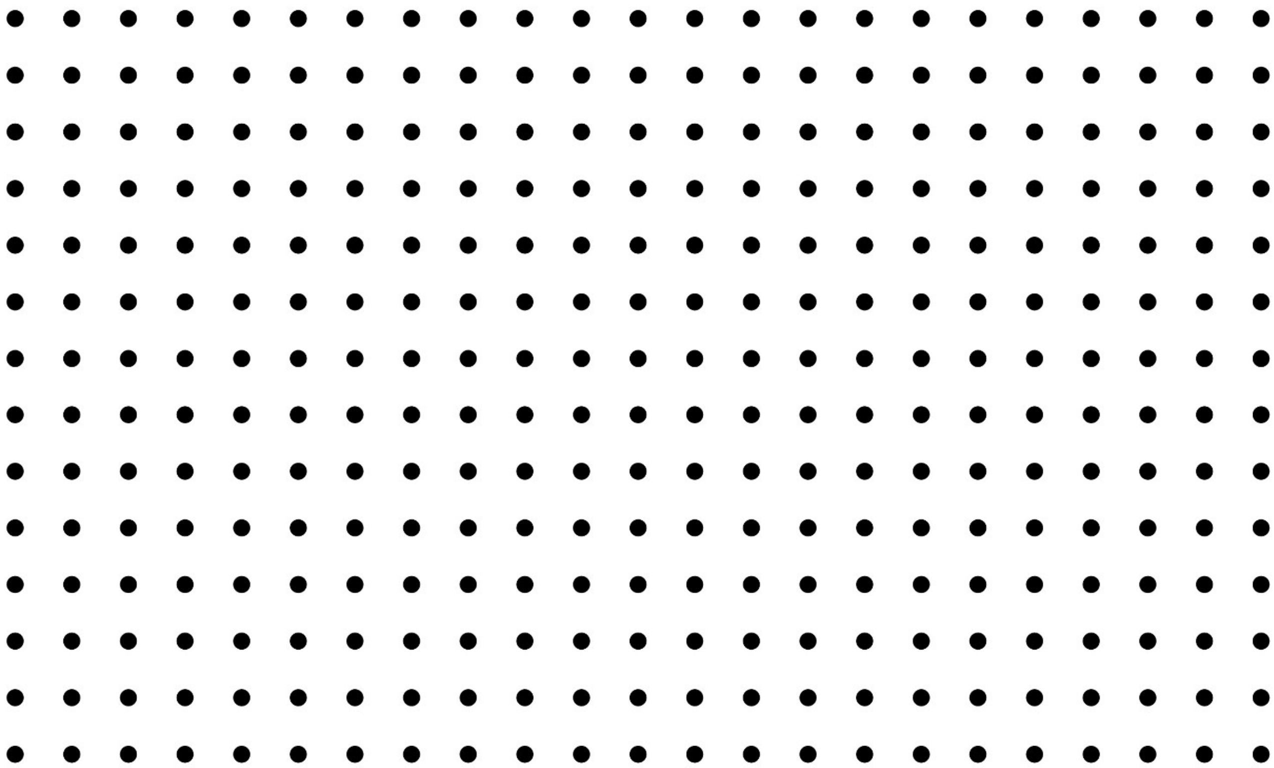
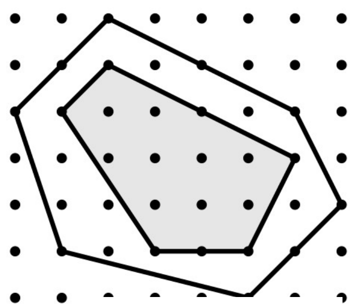


Figure 4

The relation holds for all polygons that have vertıces on the grid points. So this relation holds for all grid-polygons. Encourage the students to refer to these polygons "grid polygons" and that the relation holds for these.

In this possible that some of your students might make some shapes on the grid where Pick's Theorem does not hold. Like the shape given below.



Shape with a hole
Figure T2

This can be a good opportunity to point out the shapes where Pick's Theorem holds and where it does not.
It always works for a polygon but does not work for shapes which have holes in them, similar to the figure shown in Figure T2.

We have looked at some polygons and found an expression for their area by just counting the boundary and the interior points.

For any shape S, with ***I*** = number of grid-points in its interior and ***B*** = number of grid-points on its boundary. Let is define Pick(S) as:

$$\textbf{Pick(S)} = I + \frac{B}{2} - 1$$

From the polygons that we have drawn, we saw that for a shape S,
Pick(S) = Area(S)
This relation is called Pick's Theorem.

Now let us look at some special polygons and prove that this theorem holds for them too.

Task 6: Special cases!

In this part, we will look at some special cases. We will find some special quadrilaterals like straight squares and straight rectangles. Do you know what is a straight rectangle and square?
Let us find out.
Look at the rectangles in Figure 5, we will call Rectangles 2, 4, and 6 as straight rectangles and Rectangles 1, 3 and 5 as slanted rectangles. Note that Rectangle 6 is also a straight square and Rectangle 1 is a slanted square.

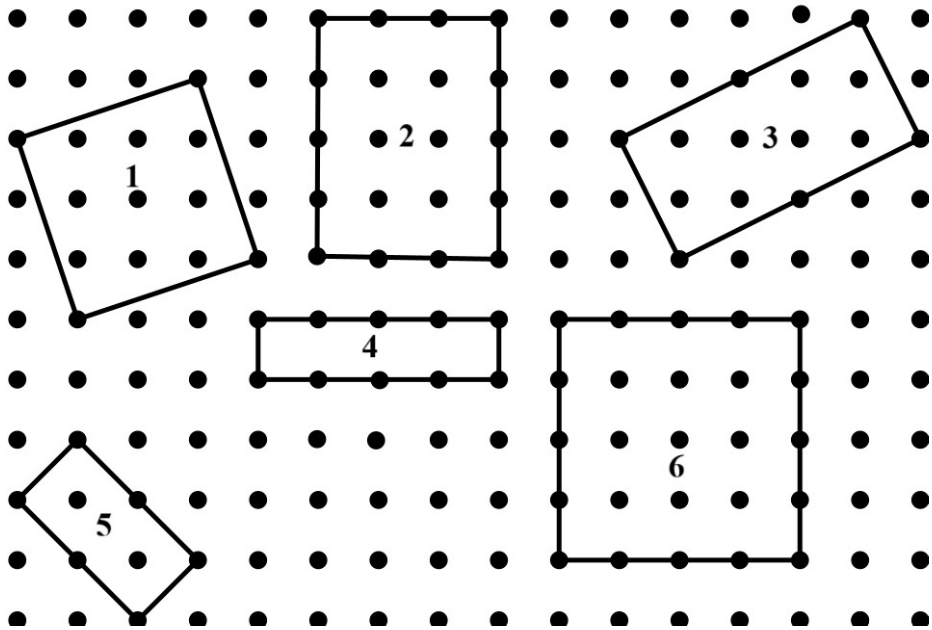


Figure 5

Let us try to prove Pick's Theorem for a straight square and a straight rectangle. What do we mean by proving Pick's Theorem for a straight square and rectangle?

We know that for a shape, S: Pick(S) = $I + \frac{B}{2} - 1$ ... Pick's Formula

So to prove Pick's Theorem, we have to count the interior points and boundary points of a straight square or a straight rectangle. Then put them in the Pick's Formula and see if it is equal to the area of a straight square and a straight rectangle which we know.

We know that for a square of side m units, the area of a square is m^2
And we know that for a rectangle of sides m and n units, the area of the rectangle is mn.

So we have to show that,
Pick(Straight Square) = m^2 and
Pick(Straight Rectangle) = mn

Let us try to prove Pick's Theorem for a straight square. But before that let us look some specific straight squares.

i. A square of side 6 units.

On a line segment of 6 units, the number of points on the line segment = (6 + 1).
Total number of points of the square = $(\_\_\_\_)^2$ = _____
We have four line segments of length 6 units as the boundary of the square.
So, the number of points on the boundary
B = __________ = 4 × ____
(Remember to check for points which have been counted twice, namely the ones
at the corners)

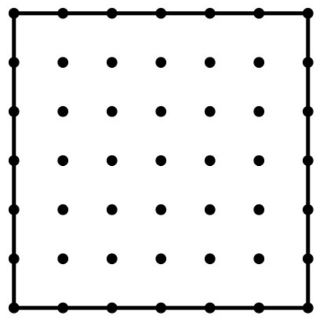


Figure 6

Also, total number of points of the square
= Number of points inside the square + Number of points on its boundary

Number of points in the interior of the square,
I = Total number of points – Points on the boundary
I = ___________ − ___ = ____
So, Pick(Square) = $I + \frac{B}{2} - 1$ = ____________.
We know that, Area(Square) = ____

So, Area(Straight Square) _____ Pick(Straight Square) (Fill in the blanks with <, > or = sign)

ii. A square of side 5 units.

On a line segment of 5 units, the number of points on the line segment
= (5 + 1).
Total number of points of the square
= Number of points inside the square + Number of points on its boundary
= $(\_\_\_\_)^2$ = _____

We have four line segments of length 5 units as the boundary of the square.
So, the number of points on the boundary
B = __________ = 4 × ____
(Remember to check for points which have been counted twice, namely the ones at the corners)

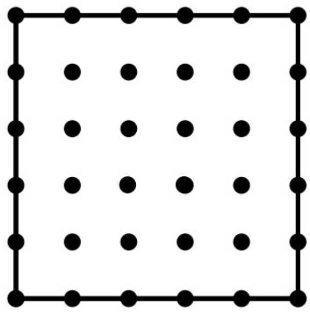


Figure 7

Number of points in the interior of the square, I =
Total number of points – Points on the boundary
I = _________________ − __________ = _________

So, Pick(Straight Square) = $I + \frac{B}{2} - 1$ = ____________.
We know that, Area(Straight Square) = _____
So, Area(Straight Square) _____ Pick(Straight Square) (Fill in the blanks with <, > or = sign)

Now let us look at a general $m \times m$ square and see how the Area(Straight Square) is related to Pick(Straight Square)

a) For a straight square of side m units

Notice: On a line segment of m units, the number of grid points on the line segment is (m + 1).

Total number of points of the square = $(m + 1)^2$ = ______________

Also, total number of points of the square
= Number of points inside the square + Number of points on its boundary

We have four such line segments as the boundary of the square.
So, the number of points on the boundary
B = ______________ = 4 × _____
(Remember to check for points which have been counted twice, namely the ones at the corners)

Number of points in the interior of the square, I =
Total number of points – Number of points on the boundary
I = ______________________ – ______________ = _____________

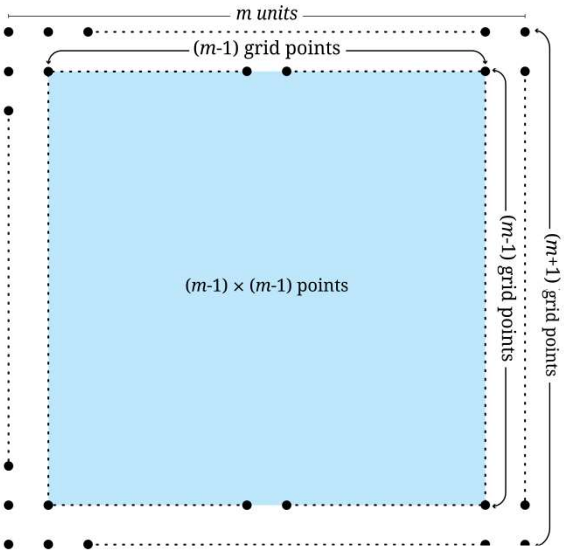


Figure 8

So, Pick (Straight Square) = $I + \frac{B}{2} - 1$ = _____________.
We know that, Area (Straight Square) = ____
So, Area (Straight Square) _____ Pick (Straight Square) (Fill in the blanks with <, > or = sign)

For a straight square of side m units,
On a line segment of m units, the number of points on the line segment = (m + 1).
Total number of points of the square = $(m + 1)^2 = m^2 + 2m + 1$
Also, total number of points of the square
= Number of points inside the square + Number of points on its boundary = $(m + 1)^2 = m^2 + 2m + 1$
We have four such line segments as the boundary of the square.
So, the number of points on the boundary
$B = 4 \times (m + 1) - 4 = 4m + 4 - 4 = 4m$
(4 points were counted twice and hence are subtracted)
Number of points in the interior of the square, I = Total number of points – Points on the boundary
$I = (m^2 + 2m + 1) - 4m = m^2 - 2m + 1$
So, Pick (Straight Square) = $I + \frac{B}{2} - 1 = m^2 - 2m + 1 + \frac{4m}{2} - 1 = m^2 - 2m + 1 + 2m - 1 = m^2$
We know that, Area (Straight Square) = m^2
So, Area (Straight Square) = Pick (Straight Square)

b) For a straight rectangle of length m units and breadth n units

On a line segment of m units, the number of points on the line segment is (m + 1).
And, on a line segment of n units, the number of points on the line segment is (n + 1).
Total number of points of the rectangle = $(m + 1)(n + 1)$ = ______________
Also, total number of points of the rectangle = Number of points inside the rectangle + Number of points on its boundary
On the boundary, there are two line segments of m units and two line segments of n units.
So, the number of points on the boundary, B = ___________

(Remember to check for points which have been counted twice, namely the ones at the corners)
Number of points in the interior of the straight rectangle, I

= Total number of points – Points on the boundary = _____ − _____

The number of points in the interior, I = ________

Pick (Rectangle) = $I + \frac{B}{2} - 1$ =______

We know that, Area (Straight Rectangle) =_______

So, Area (Straight Rectangle) _____ Pick (Straight Rectangle) (Fill in the blanks with <, > or = sign)

In the case of the straight rectangle we get that:
On a line segment of m units, the number of points on the line segment is (m + 1). On a line segment of m units, the number of points on the line segment is (m + 1).
And, on a line segment of n units, the number of points on the line segment is (n + 1).
Total number of points inside and outside this rectangle = $(m + 1)(n + 1) = mn + m + n + 1$
On the boundary, there are two line segments of m units and two line segments of n units.
So, the number of points on the boundary, $B = 2m + 2n + 4 - 4 = 2m + 2n$
(Remember to check for points which have been counted twice, namely the ones at the corners)
Number of points in the interior of the straight rectangle, I
= Total number of points – Points on the boundary
$= mn + m + n + 1 - (2m + 2n) = mn - m - n + 1$
The number of grid-points in the interior, I = ________
Pick (Straight Rectangle) = $I + \frac{B}{2} - 1 = mn - m - n + 1 + \frac{1}{2}(2m + 2n) - 1 = mn$
We know that, Area (Straight Rectangle) = mn
So, Area (Straight Rectangle) _____ Pick (Straight Rectangle)

Some students may need to consider a few specific cases before they arrive at general expressions for the number of grid-points in the interior and boundary of a general straight square, and a straight rectangle.

You can spend some time discussing with students why we are using a straight square and a straight rectangle. You can point out how counting boundary points and interior points is simple and possible in case of straight square and straight rectangles and how generalizing can be difficult in case of general squares and rectangles.

So, for a straight rectangle and a straight square, P, we proved that

Pick (P) ___________ **Area (P)** (Fill in the blanks with <, >, or = signs)

This relation is called Pick's Theorem.

So, we have proved Pick's theorem for a very special class of shapes namely straight squares and straight rectangles. We now will go on to see if Pick's Theorem is true for all shapes on the grid paper. But before that, let us go back to Bahubali and his elephants!

Task 7:

Find the area of Bahubali's garden

Students can now use Pick's Theorem to find the area of the coconut garden.
The following tasks give some hints to prove the Pick's Theorem for a general grid-polygon.

Task 8: What about any polygon?

We have proved that Pick's theorem holds for any straight square or any straight rectangle. But can we say that Pick's theorem holds for any polygon?

In the following tasks we will look at more such special cases and go on to prove the Pick's theorem for any grid polygon.

Look at the given pentagon.

a) Can you divide this pentagon into non-overlapping triangles, such that sum of area of all triangles is equal to the area of the pentagon?

(Remember: All the vertices of each triangle should be vertices of the polygon)

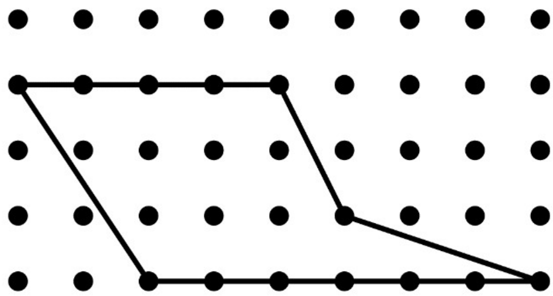


Figure 9

How many triangles did you get?

There are many ways to do this triangulation. One of the ways is shown below.

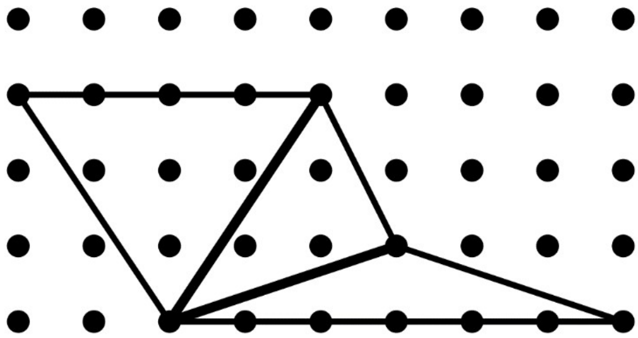
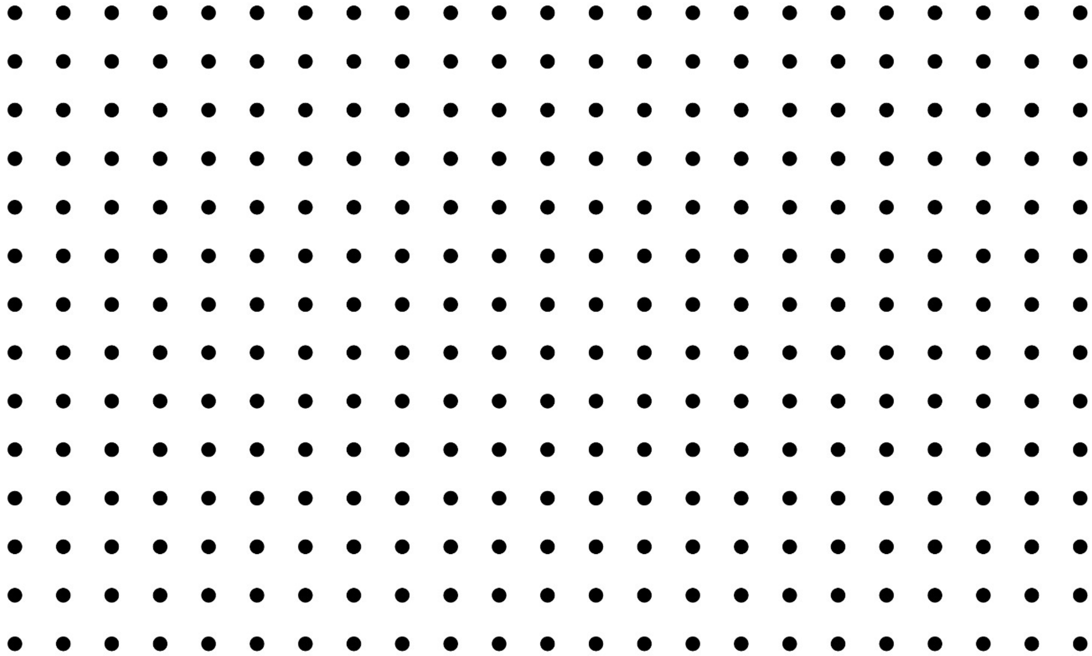


Figure T2

b) Draw more polygons on the grid paper given to you and find how many such triangles did you get in each of the polygons?



This is a good time to ask them to generalize to a polygon with *n* vertices where such a polygon can be divided into (*n* - 2) non-overlapping triangles such that all vertices of all triangles are vertices of the polygons as well. In this case the sum of areas all such triangles is always equal to the area of the polygon. But it is possible that your students will divide the shapes into more than (*n* -2) triangles, which is also fine. The important thing is to notice that any polygon can be divided completely into non-overlapping triangles.

We saw that any polygon can be divided into triangles. So in order to prove that Pick's theorem holds for any polygon, we need to prove 2 things

1) Pick's Theorem holds for any triangle,

2) Given two shapes for which the theorem holds, it also holds for the shape formed by joining these two shapes edge-to-edge without overlap

Then we can say that Pick's theorem holds for all polygons.

Spend some time on this point. Underline the point that we have broken down the problem of proving Pick's theorem for all grid polygons to two simpler problems mentioned above.

Task 9: Joining and counting!

If we put together two shapes, say Shape P and Shape Q, in such a way that they share a boundary, to form Shape R then we know that,

Area of (R) = Area (P) + Area (Q)

Look at the shapes P and Q in Figure 10. Imagine P and Q are joined to get the shape R.

Let us assume that Pick's Theorem holds for shapes P and Q. So we know

Pick P = $(I_{\mathrm{P}} + \frac{B_{\mathrm{P}}}{2} - 1)$ = Area (P) and Pick Q = $(I_{\mathrm{Q}} + \frac{B_{\mathrm{Q}}}{2} - 1)$ = Area (Q)

We have to prove that: Pick R = $(I_R + \frac{B_R}{2} - 1)$ = Area (R)

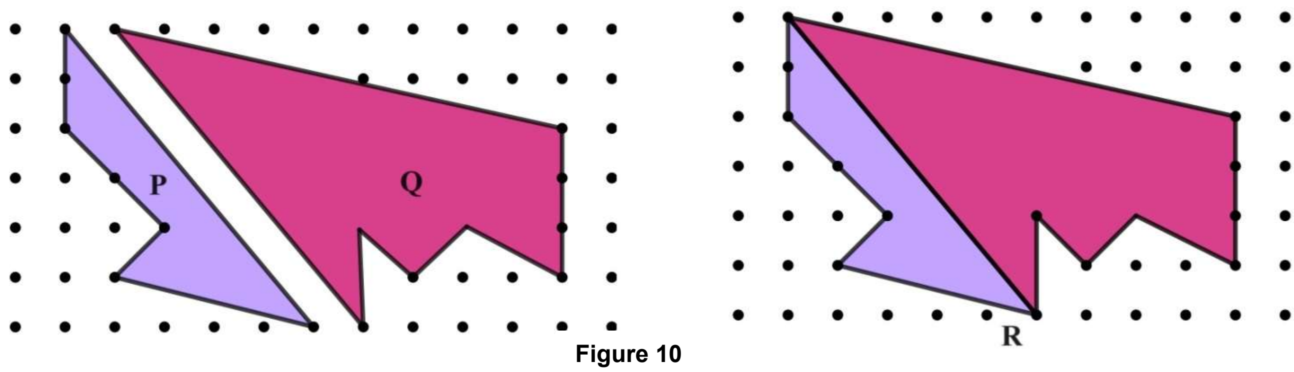


Figure 10

Let I_P, I_Q and I_R be the number of grid-points in the interior of P, Q and R respectively and B_P, B_Q and B_R be the number of grid points in the boundary of P, Q and R respectively.
Now, let us count I_R and B_R in terms of I_P, I_Q, B_P and B_Q.

Let c be the number of grid points on the common boundary of P and Q.

(a) What is the relation between the numbers of boundary points of P, Q related to the number of boundary points of R?
(b) Can you come up with an expression for I_R and B_R in terms of I_P, I_Q, B_P and B_Q?

I_R = $I_P + I_Q$ + ____ – ____, (Fill in the blanks) …… (1)

B_R = $B_P + B_Q$ – ____ + ____ (Fill in the blanks) …… (2)

(Hint: Remember the number of points of the common boundary, c will play an important role in this)

If c is the number of grid points on the common boundary,
$I_R = I_P + I_Q + c - 2$
So, we have $I_R - c + 2 = I_P + I_Q$ …… (3)
$B_R = B_P + B_Q - 2c + 2$
So, we have $B_R + 2c - 2 = B_P + B_Q$ …… (4)

Now, if we assume that Pick's Theorem holds for P and Q, then what do we get?

Area (P) = Pick (P) = __________________

Area (Q) = Pick (Q) = _________________

Now we know that Area(R) = Area (P) + Area (Q)
So,
Area (R) = Pick (P) + Pick (Q)

(Hint: Use the expressions of I_R and B_R from (1) and (2))

Area (R) = Area (P) + Area (Q) = Pick (P) + Pick (Q)

$= (I_P + \frac{B_P}{2} - 1) + (I_Q + \frac{B_Q}{2} - 1)$

$= (I_P + I_Q) + (\frac{B_P}{2} + \frac{B_Q}{2}) - 2$

$= (I_P + I_Q) + (\frac{B_P}{2} + \frac{B_Q}{2}) - 2$

Substituting (3) and (4) in the above equation,

$= (I_R - c + 2) + (\frac{B_R + 2c - 2}{2}) - 2 = (I_R - c + 2) + (\frac{B_R + 2c - 2}{2}) - 2$

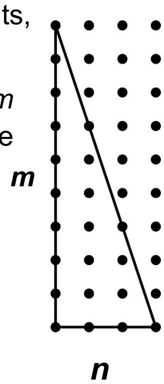
$= I_R - c + 2 + \left(\frac{B_R}{2}\right) + c - 1 - 2 = (I_R) + \left(\frac{B_R}{2}\right) - 1$

= Pick (R)

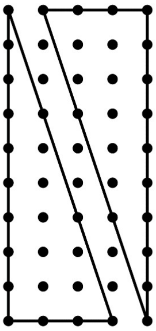
Task 10: Another special case!

In Task 9, we saw that to prove Pick's Theorem for all grid polygons we need to prove Pick's Theorem for all triangles and joining of triangles. In Task 8, we saw that Pick's theorem works for joining. So now we need to prove that Pick's Theorem holds for all triangles. But before that let us look at a very special case of triangles, namely a straight right-angle triangle of height *m* units and base *n* units, where *m* and *n* are both integers.

For a straight right-angle of height *m* units and base *n* units, (*m*, and *n* are integers)



Now we can take another congruent right-angle triangle



And join them to make a straight rectangle.

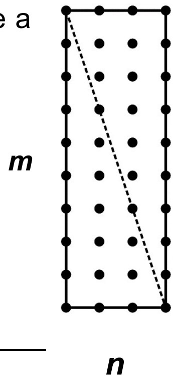


Figure 11

The straight rectangle we get is of length *m* units and breadth *n* units.

From Task 6, we know that,

The number of grid-points in the interior (I) of the straight rectangle = ________________

The number of grid-points on the boundary (B) of the straight rectangle = _____________

Area (straight rectangle) $= I + \frac{B}{2} - 1$ = ____________

The number of grid-points in the interior (I) is $(m - 1)(n - 1)$

The number of grid-points on the boundary (B) is $2(m + n)$.

Then, Pick (Rectangle) = $I + \frac{B}{2} - 1 = mn - m - n + 1 + \frac{2m + 2n}{2} = mn =$ Area of the rectangle.

Now, look at (1) and (2) from Task (8) where *c* is the number of points of the boundary

$I_P + I_Q = I_R$ – __ + ___
$B_P + B_Q = B_R$ + ___ – ___

Also, because the triangles are congruent and symmetric on the grid, we know that here, $I_P = I_Q$ and $B_P = B_Q$
(Here P and Q are the two right-angle triangles and R is the rectangle made by joining them.)

You might have to ask the students to look at their worksheets to look at the Task 8.

So, we get,
___× $I_P = I_R - c + 2$
___× $B_P = B_R + 2c - 2$

$I_R = 2 \times I_P + c - 2$
$B_R = 2 \times B_P - 2c + 2$
We also know that, Area of rectangle = ___ × Area of right-angle triangle
And, Pick (straight rectangle) = Area (straight rectangle)
So, ___ × Area (Right-Angle Triangle) = Area (Re) = $I_{\text{Re}} + \frac{B_{\text{Re}}}{2} - 1$
where, B_{Re} = Number of boundary points of the straight rectangle and
I_{Re} = Number of interior points of the straight rectangle
Using the equations given above,
___ × Area of (Right-angle triangle) = Pick (R) = ____ × Pick(P)

So, Area (Right-angle triangle) = Pick (Right-angle triangle)

Pick (Re) = Area (Re)
2 × Area(Right-Angle Triangle) = Area (Re) = $I_{\text{Re}} + \frac{B_{\text{Re}}}{2} - 1$
2 × Area(Right-Angle Triangle) = $2I_{\text{Rt}} + c - 2 + \frac{1}{2}(2B_{\text{Rt}} - 2c + 2) - 1$
(where, B_{Rt} =Number of boundary points of the right-angle triangle and
I_{Rt} = Number of interior points of the right-angle triangle
$= 2I_{\text{Rt}} + c - 2 + B_{\text{Rt}} - c + 1 - 1$
$= 2I_{\text{Rt}} + B_{\text{Rt}} - 2$
$= 2(I_{\text{Rt}} + \frac{B_{\text{Rt}}}{2} - 1)$
= 2 × Pick (Rt)
So, Area(Rt) = Pick(Rt)

Task 11: Triangle inside a rectangle?

Till now we have checked that Pick's Theorem holds for straight squares, straight rectangles, and straight right-angle triangles. We also looked at how Pick's Theorem holds even if you join two shapes edge-to-edge without overlap.
Look at Figure 12 given below and find out what else do you need to show to prove Pick's Theorem for all polygons.

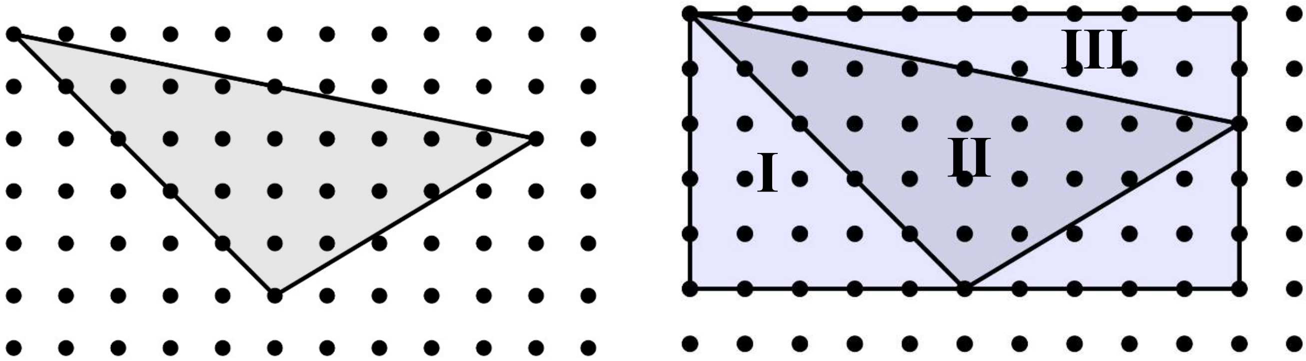


Figure 12

Story so far:
In the Learning Unit, we started with trying to find a relation between the area of a grid polygon and its boundary and interior points.

Once we got the relation, we set out to prove it. For proving the relation for a general grid polygon, we started by proving the relation for any straight rectangle and a straight square. Then we noticed that any grid polygon can be divided into non-overlapping triangles such that sum of areas of these triangles is equal to the area of the polygon.

Now our goal became proving the relation of triangle, But there was a small issue. When you join two shapes to make another one, some boundary points get counted twice and some others become interior points. So we had to show that in case of joining two shapes where the relation holds, the relation holds for the larger shape too.

Now all we need to do was to prove that relation holds for any triangle. Figure 12 tell us how we can prove it.

From the given diagram, one can see that any triangle can be enclosed in a straight rectangle and the remaining shapes in the rectangle will be straight right-angle triangles.

In the above tasks, we have looked at Pick's Theorem for straight rectangles, straight right-angle triangles and we also checked that Pick's Theorem holds when you join two shapes.
Now, we need to prove that,
Pick(Rectangle) = Pick(triangle we need) + Pick (Right triangle I) + Pick(Right triangle II) + Pick (Right triangle III) and rearrange to get,
Pick(triangle we need) = Pick(Rectangle) - Pick (Right triangle I) - Pick(Right triangle II) - Pick (Right triangle III)
An outline of the proof can be found here: https://nrich.maths.org/5441
or here https://en.wikipedia.org/wiki/Pick%27s_theorem

In this unit in order to motivate the theorem, we have considered triangles which have no interior points and came up with the relation for this special case and then generalised it to other grid-polygons. In proving the theorem also, we have proved it for the special cases of a straight rectangle and a straight square and provided hints for the general proof. We started with a special case and moved to increasingly general cases.

This is just one of the ways of motivating the theorem and proof. You may want to consider other ways like

*Motivating the theorem itself through special polygons - rectangle and square
* Drawing grid-polygons with increasing number of interior points – starting from say 0, through 1, 2, 3, …. and observing the relation between the area and the number of grid-points on the boundary and interior
* Observe the relation between area and the number of grid-points on the boundary and interior of general grid-polygons
* Or any other track that you may find comfortable.

The key idea is to have students observe and find patterns, come up with examples to verify or refute this conjecture and go on to prove it.

It is also a good idea to think of extending the task – For example, some of the questions that could be explored here are
* Would the theorem still hold if there are 'holes' in the polygon? How would one need to modify the theorem (if possible) to accommodate this case?
* Would the theorem still hold if some of the boundaries of the shape are curved? How would one need to modify the theorem (if possible) to accommodate this case?
* In case there are some curved boundaries, would it be possible to 'cover' the shape with a grid-polygon and thus come up with upper and lower bounds for its area?

Notice that we are considering more and more general cases of shapes on a grid and exploring if the theorem holds for these shapes. This is also an aspect of mathematics that could be conveyed to students through this unit. Invite students to propose variations and ask their own questions to extend the task. Even if the solutions to all extensions proposed are not found out, the exercise of thinking through these variations may itself be valuable.

References

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http://www.geometer.org/mathcircles/pick.pdf
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