Counting Coins, Bees and Dominoes

Overview

The LU presents students with a set of problems which would lead them to a very interesting sequence of numbers called the Fibonacci sequence or Fibonacci numbers. While solving these problems students will also learn to justify the patterns they observe. After the initial introduction to this sequence, students will explore many activities connected to these numbers. They will construct the Fibonacci spiral, observe patterns in this sequence of numbers and learn a bit about the history of these numbers.

Thus, this LU aims at encouraging students to make connections, observe patterns and verify them in the context of the Fibonacci numbers.

Minimum Time Required:

5 sessions of 40 min

Type of Learning Unit

Classroom

Learning Objectives

Developing problem solving abilities
Justifying the patterns observed
Making connections
Observing and verifying patterns

Materials required
Grid papers, Scissors. Graph paper

Introduction

Let us start by solving some problems.

Part 1: Let us count!

Task 1.1:
Solve the problems given below:

1. A ticket-vending machine accepts only Re. 1 and Rs. 2 coins. It gives out tickets of different values.

For example, if you want a Re 1 ticket then you can put the coins only in one way, namely
Step 1: Put a 1 Re coin.

But there can be different ways of getting a ticket of the same value, like if you want a Rs 2 ticket then you have two ways of getting it.

Way 1. *Step 1*: Put a 1 Re coin, *Step 2*: Put another 1 Re coin
Way 2. *Step 1*: Put a Rs 2 coin
These ways can be written as Way 1: (1, 1), and Way 2: (2).

In the case when you want a Rs 3 ticket, what are the different ways you can get it?
Way 1. *Step 1*: Put a Re 1 coin, *Step 2*: Put a Re 1 coin, *Step 3*: Put a Re 1 coin, thus (1, 1, 1).
Way 2. *Step 1*: Put a Rs 2 coin, *Step 2*: Put a Re 1 coin, thus (2, 1).
Way 3. *Step 1*: Put a Re 1 coin, *Step 2*: Put a Rs 2 coin, thus (1, 2).

Are there any more ways you can get a Rs 3 ticket?

Fill the entries in the table with various ways in which you can get tickets with different values.

Cost of ticket	**Ways**	**Number of ways**
1	(1)	1
2	(1, 1), (2)	2
3	(1,1,1), (2,1), (1,2),	3
4		
5		
6		

Table 1

2. All of us must have seen a honeybee hive. Honeybees make those hives to store nectar or honey. In a honeybee hive there are three types of bees; the queen bee, worker bees and the drone.

The queen is a female bee who produces eggs. The worker bees are also females but they do not produce any eggs. The drones are the male bees who do not do any work.

All female bees have 2 parents, a male and a female. And the drones have only one parent, a female bee.

Here we do not follow the convention of Family Trees that *parents appear above their children.* In this family tree, the latest generation is at the top and the lower we go, the older the bees get. In this tree all the *ancestors* of the bee are below the bee.

The family tree of the male bee will look something like this:

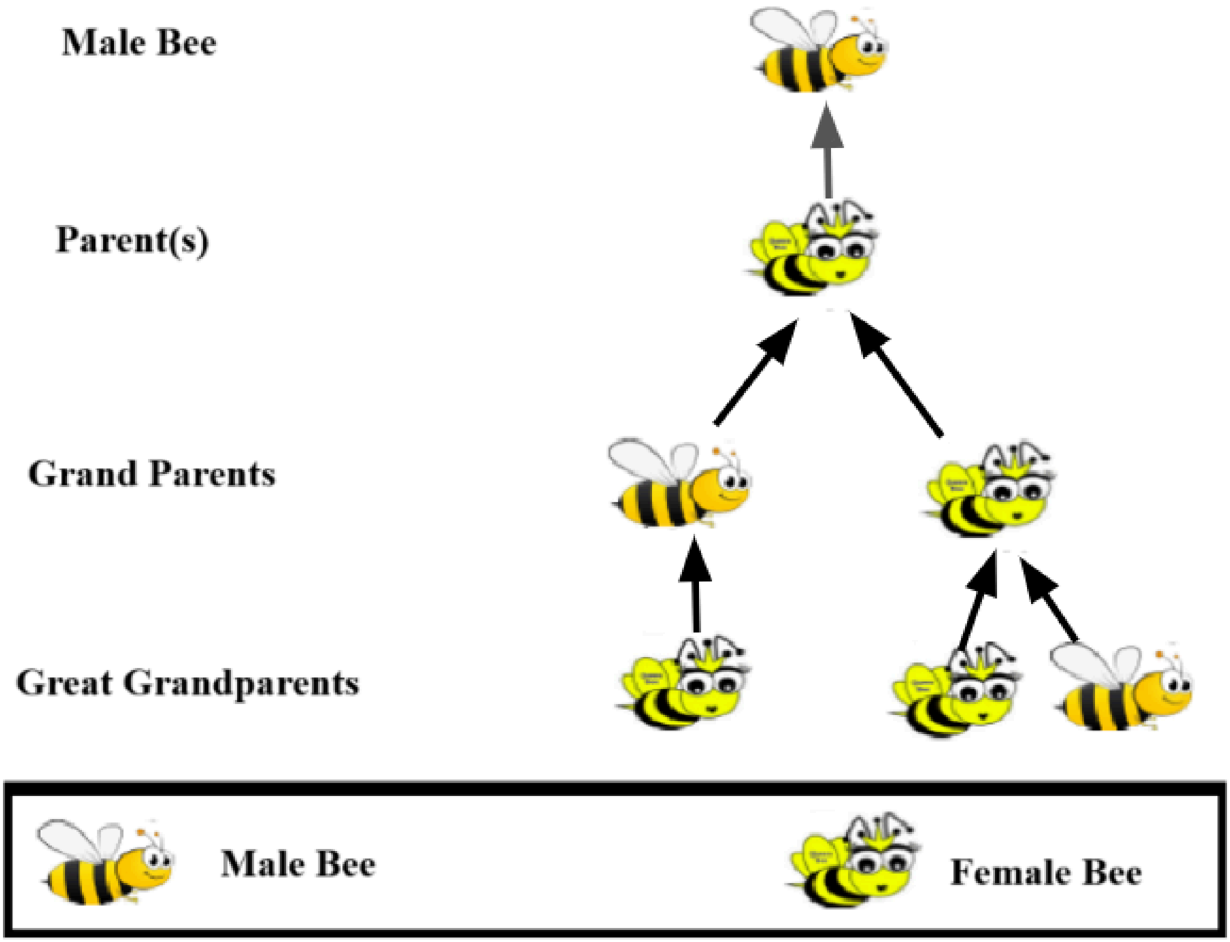


Figure 1: Family Tree of a male bee

Extend the family tree of male bees and draw one for female bees. And then complete the table given below.

No.	Parent	Grandparents	Great-Grand-Parents	GT-Great-Grand-Parents	GT-GT-GT-Grand-Parents	GT-GT-GT-GT-Grand-Parents	GT--GT-GT-GT-GT-Grand-Parents
Of male bee	1	2	3				
Of female bee	2	3					

Table 2

3. Suppose we have unit squares arranged in a $2 \times n$ grid. See the figures below for examples.

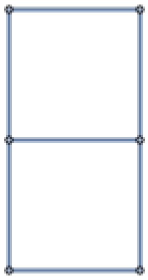


Figure 2: 2×1 grid

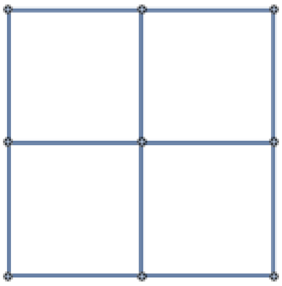


Figure 3: 2×2 grid

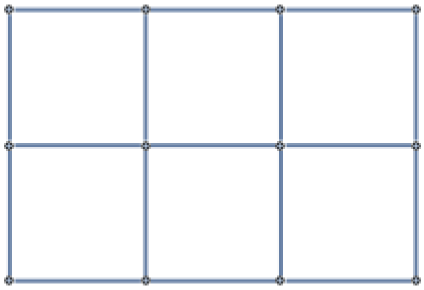


Figure 4: 2×3 grid

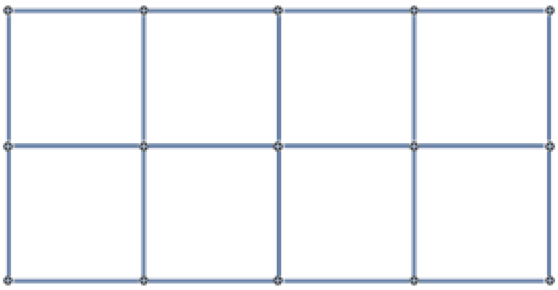


Figure 5: 2×4 grid

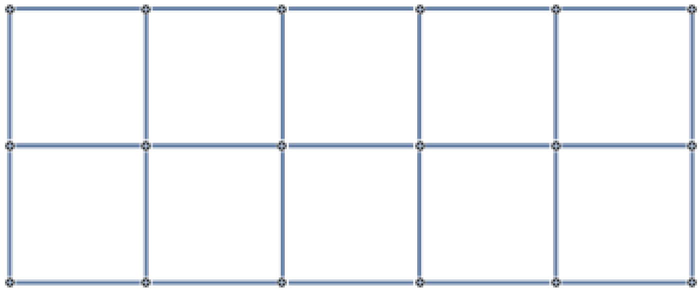


Figure 6: 2×5 grid

A **domino** is a collection of two adjacent squares. There are two possible dominoes, **vertical** or **horizontal**, as shown below.

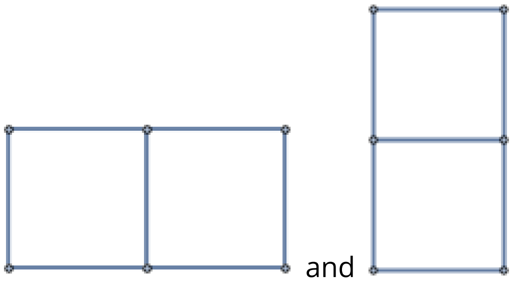


Figure 7: Two possible dominoes

How many ways are there of completely filling this $2 \times n$ grid with no overlaps. Use the dominoes given in the Activity Sheet (I) at the end of this unit. Cut the dominoes given on the sheet and find all the ways of tiling them on the rectangles in Figure 1, 2, 3 and 4 and fill the table given below.

Value of n	1	2	3	4	5	6
Number of tilings						

Table 3

After you have solved all three questions, discuss your answers with your friends. Did you solve all the problems till the end or you found a pattern and predicted your answer from it? How can you be sure that your prediction is correct?

We saw that these numbers come in different situations. Maybe that's why mathematicians find them interesting.

The sequence of numbers you got while solving the three problems in the previous part are called **Fibonacci Numbers or Fibonacci Sequence.** These numbers appear unexpectedly in mathematics, so much so that there is an entire journal dedicated to their study, the *Fibonacci Quarterly*.

When the students solve this worksheet, they will realise that the sequence of numbers that they got for all the three problems are the same. Bring to their notice the relationship between the numbers in this sequence, namely that 'Sum of any two consecutive numbers in the sequence gives you the next number in the sequence. Once the students articulate this pattern, you can ask them their reasons why they do not feel the need to actually check all the answers by counting and predict the answer by just looking at the patterns they have got. Can they figure out why in each of the situations, the final answer would be the sum of the previous two answers. Why can one predict the answers to the questions without actually calculating the answers?

For example, in the ticket vending machine problem, the number of ways you can make for Rs $n + 1$, is equal to the number of ways you can make for Rs n + the number of ways you can make for Rs $(n - 1)$ ticket. Let us look at an example and see why this would happen for every n.
Now we have to get a Rs 10 ticket and the machine accepts only Re 1 and Rs 2 coins.

Say if in Step 1, you put a Re 1 coin. So now you have to insert Rs 9 more. So the total number of ways in which you can make Rs 10 when you start by putting Re 1 coin is equal to the number of ways in which you can get a ticket of Rs 9. Notice that there cannot be any other combinations if you have put a Re 1 coin in the beginning. Because then that would mean that there is an extra way to make Rs 9 which was not listed before. Similarly, when you start by putting a Rs 2 coin, then the number of ways you will have will be the same as the number of ways you can make Rs 8..
So the number of ways you can get a Rs 10 ticket is the sum of the number of ways to get a ticket of Rs 9 and the number of ways to get a ticket of Rs 8.

In generalized terms one can say the following:
"To get a ticket of Rs n you have to start with putting a Re 1 coin or a Rs 2 coin. If you start with Re 1, then the number of ways you can make is equal to the number of ways you can make Rs $(n - 1)$. Similarly, if you start with Rs 2, then the number of ways you can make is equal to the number of ways you can make Rs $(n - 2)$.
Hence, if F_n is the total number of ways to get a ticket of Rs *n*, then

$$F_{n+2} = F_{n+1} + F_n$$

Similarly, in the case of the honey bee problem, we count bees in every generation. One thing to remember is that if F_n is the number of bees in the n^{th} generation then $F_{(n-1)}$ is the number of bees in the generation after n^{th} generation, i.e. offsprings of the bees in the n^{th} generation.
Let, Fe_n = Number of female bees in the n^{th} generation and M_n = Number of male bees in the n^{th} generation

So, $$F_n = Fe_n + M_n$$

Now, the number of female bees in the n^{th} generation are all mothers of all the bees in the $(n\text{-}1)^{th}$ generation because every bee will have only one offspring.
And all the male bees of the n^{th} generation are fathers of female bees in the $(n\text{-}1)^{th}$ generation because only female bees have a male parent.. So we get,

$Fe_n = F_{(n-1)}$ and $M_n = Fe_{(n-1)}$

So we get, $F_n = F_{(n-1)} + Fe_{(n-1)}$

Now, using similar argument as before we get, $Fe_{(n-1)} = F_{(n-2)}$

So, $F_n = F_{(n-1)} + F_{(n-2)}$

Encourage your students to find similar reasoning for the remaining problems.

History

The oldest known appearance of the Fibonacci numbers is in India. It is known that Pingala (roughly 450–200 BCE), the ancient Indian mathematician and poet, studied them in the context of Sanskrit poetry. Later, Virahanka (c. 700 CE) wrote a detailed study about these numbers. Though a lot of Virahanka's work is lost, one can find references of his work in the later work by Gopala (c. 1135 CE).

These numbers are named after the mathematician, Leonardo Fibonacci (son of Bonacci), also known as Leonardo of Pisa. Fibonacci talked about these numbers in his book, *Liber Abaci*, in the growth of populations of rabbits, which was published in 1202. Do try to find out about these mathematicians from the internet or your library.

Part 2: Fibonacci Spiral and Golden Ratio

Task 2.1: Constructing the Fibonacci Spiral

Another term that is often mentioned in association to Fibonacci numbers is the '*Fibonacci Spiral*'.

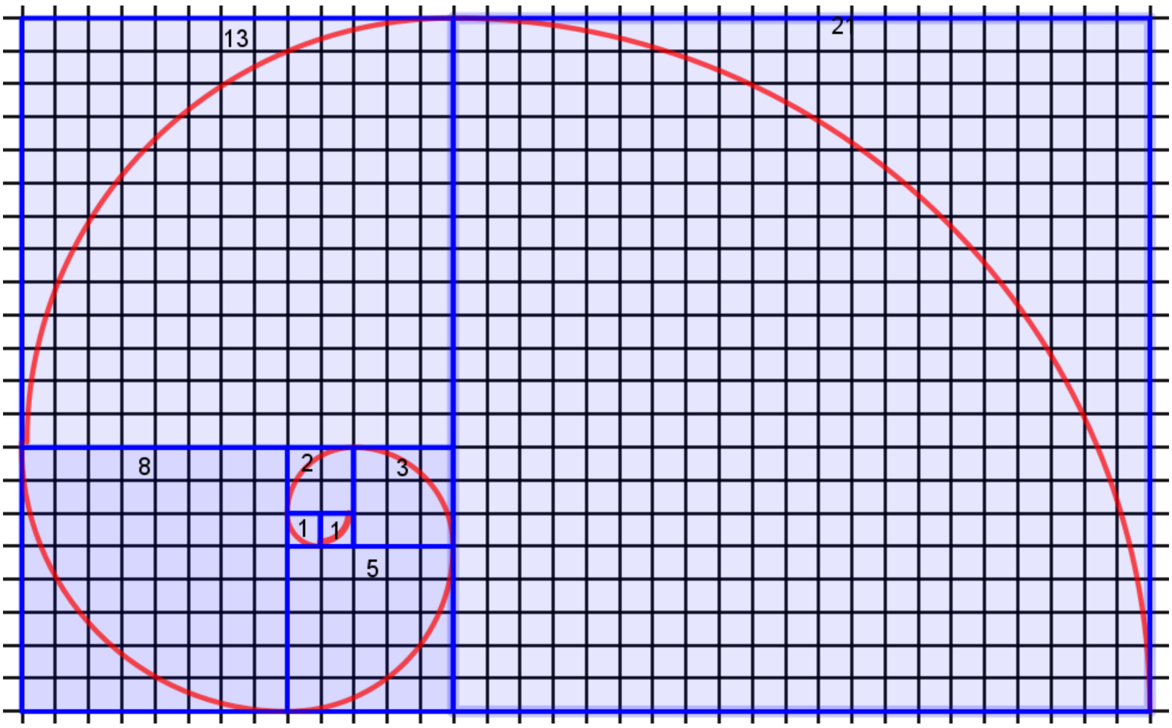


Figure 8: Fibonacci Spiral

Let us construct this spiral and see why it is associated with the Fibonacci numbers.
Start with a square of side 1 unit, in the middle of the grid paper provided to you as shown in Figure 9.

1st Fibonacci Number = 1

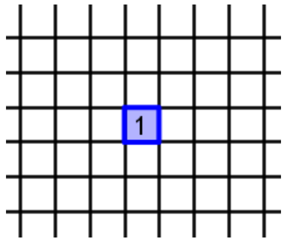


Figure 9

Then on the left of this square, draw another square of side 1 unit as shown in Figure 10.

2nd Fibonacci Number = 1

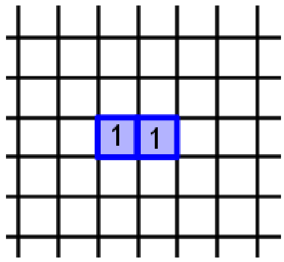


Figure 10

Now on top of these two squares draw a square of side 2 units such that it shares a side with both the above squares as shown in Figure 11.

3rd Fibonacci Number = 2

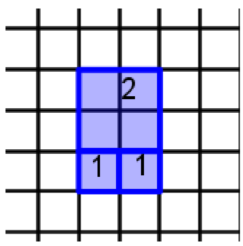


Figure 11

Now the combined figure is a rectangle. What is the height of this rectangle? ____________ units
What is its breadth? ________________ units

On the right of and adjoining this rectangle, draw a square whose side is equal to the height of this rectangle. Now you get another rectangle. What is the height of this rectangle? ____________ units
What is its breadth? ________________ units

Now draw another square below the new rectangle such that its side is equal to the longer side of the rectangle.

Continue this process, left, top, right and bottom till you get a rectangle with one of the sides as 34 units.

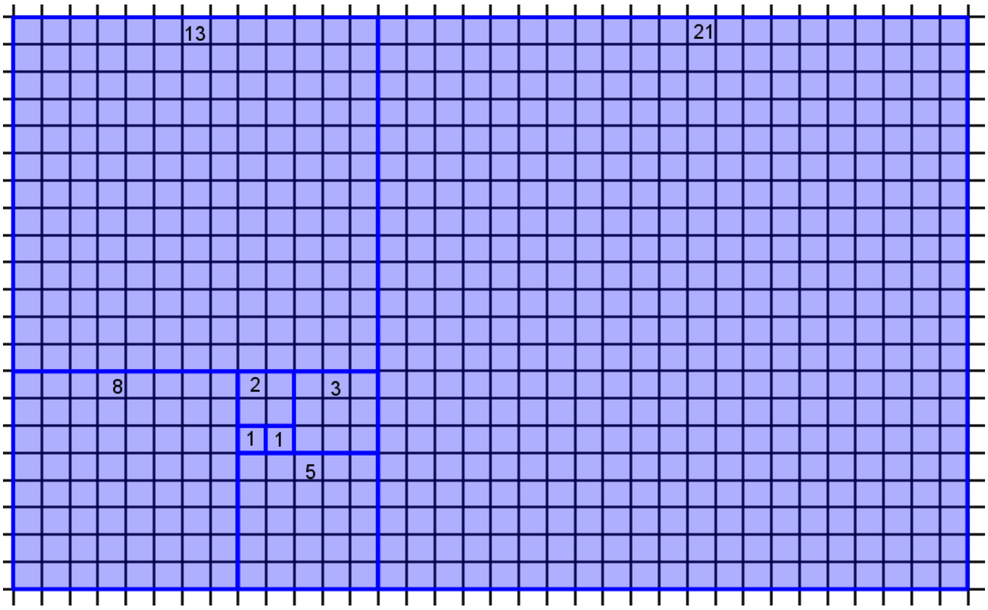


Figure 12

Now draw a quarter circle in the first square, it will look like this.

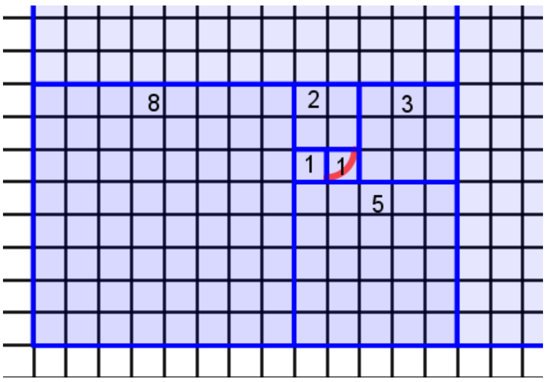


Figure 13

Continue the above arc and draw another quarter circle in the next unit square to get a semicircle.

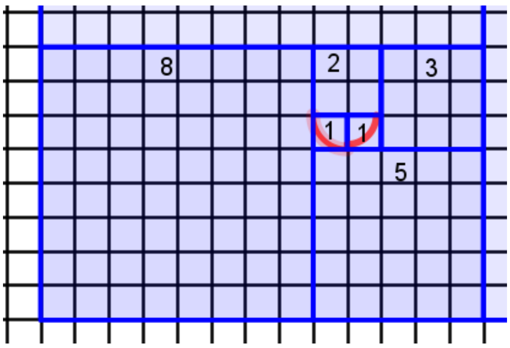


Figure 14

Continue this process of drawing quarter circles in the same direction as you drew the squares. Finally, you should get a fibonacci spiral similar to Figure 8.

This is called the **Fibonacci Spiral**. Spirals very similar to the Fibonacci Spiral occur widely in nature in various settings, e.g., shapes of galaxies, cyclones and snail shells. Try to get more information about this spiral from your library or the internet.

Figure 15

Apart from the fact that they can be seen in nature, there are other reasons why mathematicians find these numbers interesting. In the next part of this unit let us see why.

Task 2.2: Finding the Golden Ratio

Find the ratio between two consecutive Fibonacci numbers. You can use a spreadsheet or an Excel sheet for this. Notice what happens to the ratio as n increases. Your table should look like this.

n	F_n	$\frac{F_n}{F_{n-1}}$
1	1	
2	1	1.000
3	2	2.000
4	3	1.500

5	5	1.667
6	8	1.600
7	13	1.625
8	21	1.615
9	34	1.619
10	55	1.618
11	89	1.618
12	144	1.618
13	233	1.618
14	377	1.618
15	610	1.618
16	987	1.618
17	1597	1.618
18	2584	1.618
19	4181	1.618
20	6765	1.618

Table 4

As n increases, you will notice that the ratio starts coming very close to a particular number. This number is called the **Golden Ratio**. This ratio also has very interesting properties and can be seen in lots of different places in art and architecture. If you extend the table, your table should look like this.
Hence from the table we find that the Golden Ratio is approximately equal to 1.618.

Part 3: Patterns of Fibonacci numbers

Task 3.1: For different values of n, add the first n Fibonacci numbers. Does it look close to another Fibonacci number?

Ask them to notice that the sum of the first n Fibonacci numbers is one less than F_{n+2}. For example, 1+1+2+3 = 7, which is one less that $F_6 = 8$.

Task 3.2: Missing Area?

In the Activity sheet (II) given at the end of the unit, you will find two squares. Find the area of the square no. 1 and then cut it (on the dotted lines) and arrange the four pieces to form a rectangle (which is not a square). What is the area of the new rectangle?

Similarly, first find the area of square no. 2 and then cut it (on the dotted line) and arrange the four pieces to form a rectangle (which is not a square). What is the area of this new rectangle?

Observe the lengths of the squares given to you and the lengths of the rectangles you got. Have you seen these numbers before?

In the case of a square with side length 5 units, you get a rectangle of area 24 sq. units. Hence you 'miss' 1 sq. unit area. In the case of a square with side length 8 units, you get a rectangle of area 65 sq. units. Hence you 'gain' 1 sq. unit area. If you look at the numbers carefully, you will notice that all the numbers involved in this activity are Fibonacci numbers. If your students do see that all the lengths involved in this activity are Fibonacci numbers, discuss it in the class.

Let us see why this happens:

Complete the following table:

No	Fibonacci number	Square of the Fibonacci number	Product of previous number and next number
n	F_n	F_n^2	$F_{n-1} \times F_{n+1}$
1	1	--	--
2	1	$1^2 = 1$	1 x 2 = 2
3	2	$2^2 = 4$	1 x 3 =
4	3		
5	5		
6	8		
7	13		
8	21		

Table 5

Can you see any pattern emerging from the table above?

When comparing the value of the square of the Fibonacci number (F_n^2) with the product of the previous number and the next number (F_{n-1} x F_{n+1}), we will see that there is always a difference of 1.

So, if a square of side F_n units is cut to form a rectangle of sides F_{n-1} units and F_{n+1} units then their areas will differ by 1 sq.unit. This is clearly not possible according to the law of conservation of area and is an optical illusion. To prove it, observe the slope of the edges which meet at the joint. There is a gap between the edges which make up for the extra 1 sq.unit. Another way to find the 'missing area' is to use a graph paper to draw the square and then cut it to make the rectangle.

The missing area puzzle can be made from any square whose side is a Fibonacci number greater than or equal to 3.

Task 3.3: More Patterns in the Fibonacci Sequence

1. Look at the numbers given below and see if you can get many more patterns from these numbers. You can also extend the pattern if you want.

1, 1, 2, 3, 5, 8, 13, 21, 34,

2. If we write the Fibonacci sequence as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	...

Table 6

$F_3 = 2$. Now, what can you say about every 3rd number in this series, ($F_6, F_9, F_{12}, \ldots$)
$F_4 = 3$. Now, what can you say about every 4th number in this series, ($F_8, F_{12}, F_{16}, \ldots$)
$F_5 = 5$. Now, what can you say about every 5th number in this series, ($F_{10}, F_{15}, F_{20}, \ldots$)

Do you see any patterns from this table?

Extend the table and check if your patterns for other Fibonacci numbers too. You can also use a spreadsheet or an Excel sheet to do this.

3. If we square the numbers in Fibonacci series and make a new series by adding the consecutive squares, what do we get?

Observe the table carefully and discuss the pattern you got with your classmates.

F_n	1	1	2	3	5	8	13	21
F_n^2	1	1	4	9	25	64	169	441
$F_n^2+F_{n+1}^2$	2	5	13					

Table 7

4. If we add the squares of the first n Fibonacci numbers, what do we get?
Observe the table carefully and discuss the pattern you got with your classmates.

n	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21
F_{n+1}	1	2	3	5	8	13	21	34
F_n^2	1	1	4	9	25	64	169	441
$F_1^2+\cdots+F_n^2$	1	2	6	15	40	104	273	

Table 8

Though the numbers in the last row are not Fibonacci numbers, if you can factorise them such that the two factors are consecutive Fibonacci numbers.

For example:

1 = 1 x 1

2 = 1 x 2

6 = 2 x 3

15 = 3 x 5 ...

This relationship can be seen from the Fibonacci Spiral

The numbers in black are side lengths of the squares and the numbers in red are areas of the squares.

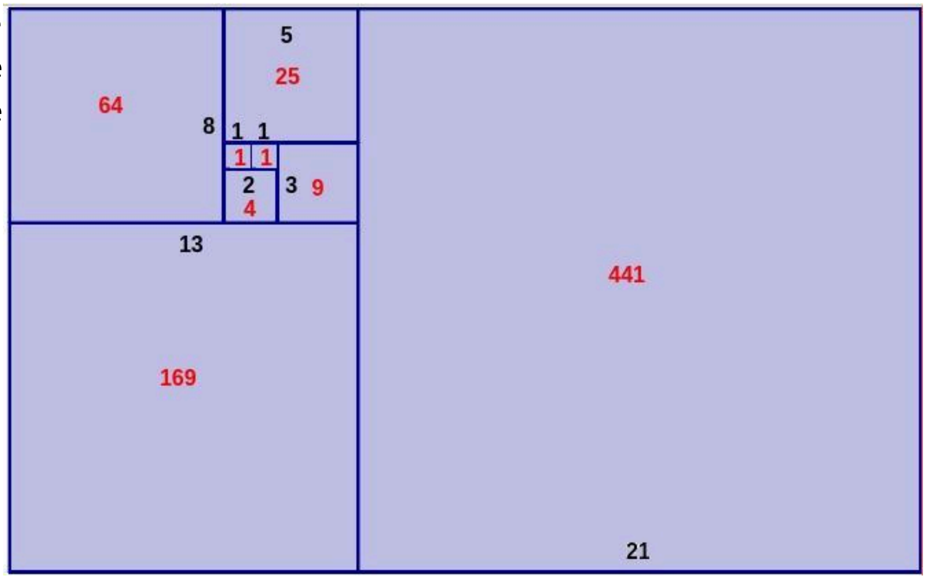


Figure T2

Now, the sum of squares of the first three Fibonacci numbers is 1, 1, and 4. The squares in the diagrams with areas 1 sq. units, 1 sq. units, and 4 sq. units together form a rectangle of sides 2 units and 3 units respectively by construction. You can check if this happens when you look at the first 4 squares or the first five squares.

Since the pattern continues, the sum of squares of all the Fibonacci numbers till F_n will be the product of F_n and F_{n+1}. You might have to ask your students to draw the squares and shade them, for them to understand the reason behind this pattern.

Some more patterns:

1. Take 4 consecutive Fibonacci numbers. If you take the difference between the product of the first and the fourth, and the second and the third, namely $F_n \times F_{n+3} - F_{n+1} \times F_{n+2}$, you will find that it is always ±1.
2. If the sum of any 10 consecutive Fibonacci numbers is taken, then it is always 11 times that of the 7th number of the 10 numbers, namely $\sum_{i=n+1}^{n+10} F_i = 11F_{n+7}$.

Extended Task

Part 4: A formula for Fibonacci Numbers

We see that Fibonacci numbers are very important. But the recursive way to get them is very tedious. To get the 100th Fibonacci number it would be necessary to find the 99th and 98th numbers. Wouldn't it be nice to have a simple form for all Fibonacci numbers?

Look at the expression given below:

$$F_n = \frac{\left(\frac{(1+\sqrt{5}}{2}\right)^n - \left(\frac{(1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$$

Substitute $n = 1, 2, ...$ and check. What did you get? Do you think this will give us a Fibonacci number? Use a spreadsheet and check what numbers you get for different values of n.

The above expression is called **Binet's formula**, named after the French mathematician Jacques Binet (1786 – 1856).

You can also use a spreadsheet to check that for every *n*, the above formula gives a Fibonacci number. The steps are as follows:
Step 1: In column A, enter 0 in cell A2 and enter the formula =A2 + 1 in cell A3. Drag the cell A3 (till A22) to generate the stage numbers till 20. This column will represent the stage numbers *n*.
Step 2: In column B, we enter the formula
"= (((1+SQRT(5))^A2/2^A2)-((1-SQRT(5))^A2)/(2^A2))/SQRT(5)" in cell B2. Drag the cell B2 (till B22) to generate the stage numbers till 20. This column will represent the stage numbers F_n.
Show it to the students.

You will notice that for each natural number you will get a corresponding Fibonacci number. Do discuss with the students how this form is very deceptive and how though it looks like an irrational expression it gives us a natural number.

So, now we know how to find Fibonacci numbers. Try to find more interesting patterns in these numbers.

Part 5: Taj Mahal and it's Golden Rectangles

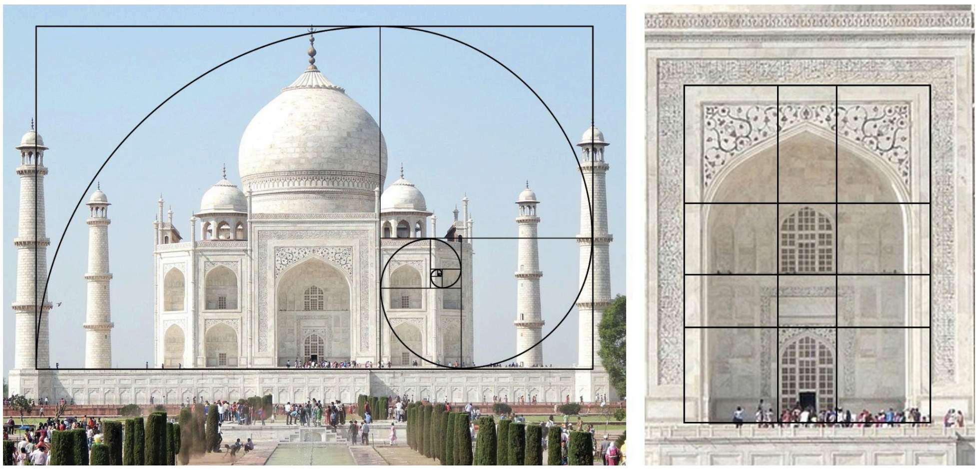


Figure 16

A rectangle whose side lengths are in the golden ratio, that is, 1:1.618 is called a golden rectangle. One characteristic that distinguishes this shape apart is that when a square chunk is taken out, what's left over is another golden rectangle, meaning it has the same proportions as the initial one.

Indian architecture has long made extensive use of the Fibonacci sequence. The Taj Mahal, one of the seven wonders of the world, follows this pattern. The Taj Mahal is not a perfect golden rectangle, but its overall proportions are close to the golden ratio.

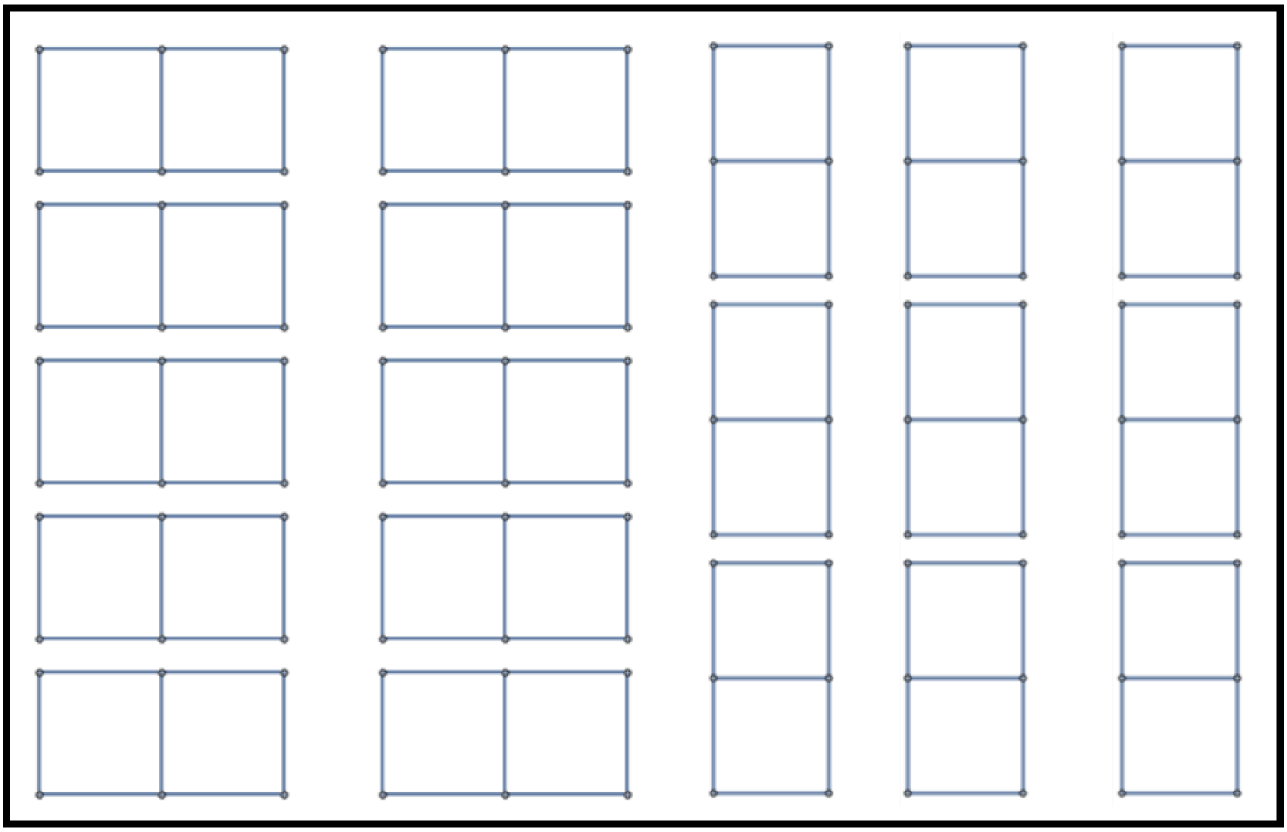
Are there any other golden shapes associated with the Fibonacci spiral or golden ratio?

Suggested resources

1. Wikipedia article on the Fibonacci sequence, https://en.wikipedia.org/wiki/Fibonacci_sequence
2. Youtube video on "Fibonacci sequence in nature", https://youtu.be/nt2OlMAJj6o
3. Youtube video on "Nature by Numbers | The Golden Ratio and Fibonacci Numbers", https://youtu.be/me6Dnl2DOtM

Activity Sheet (I)

Cut dominoes from the column on the right to tile the grids on the left



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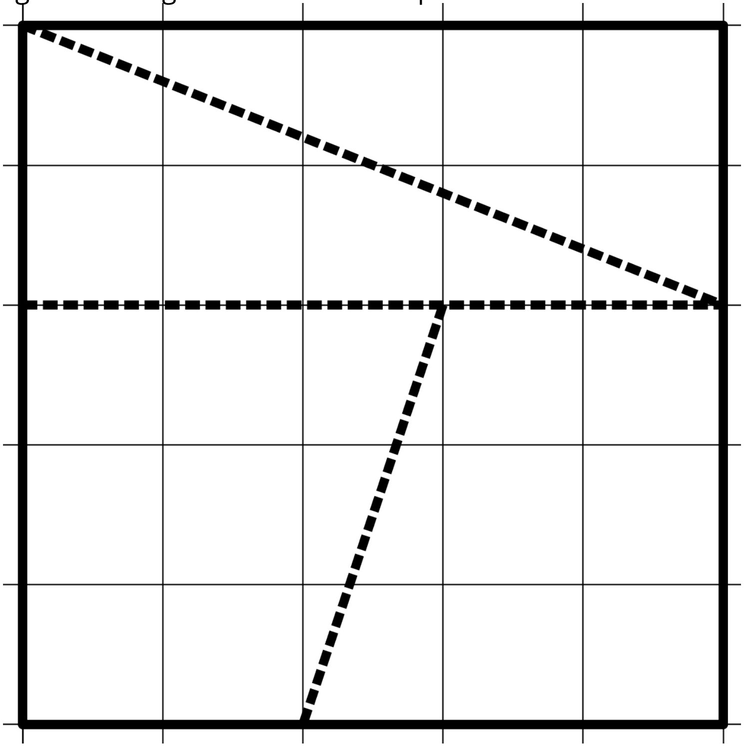
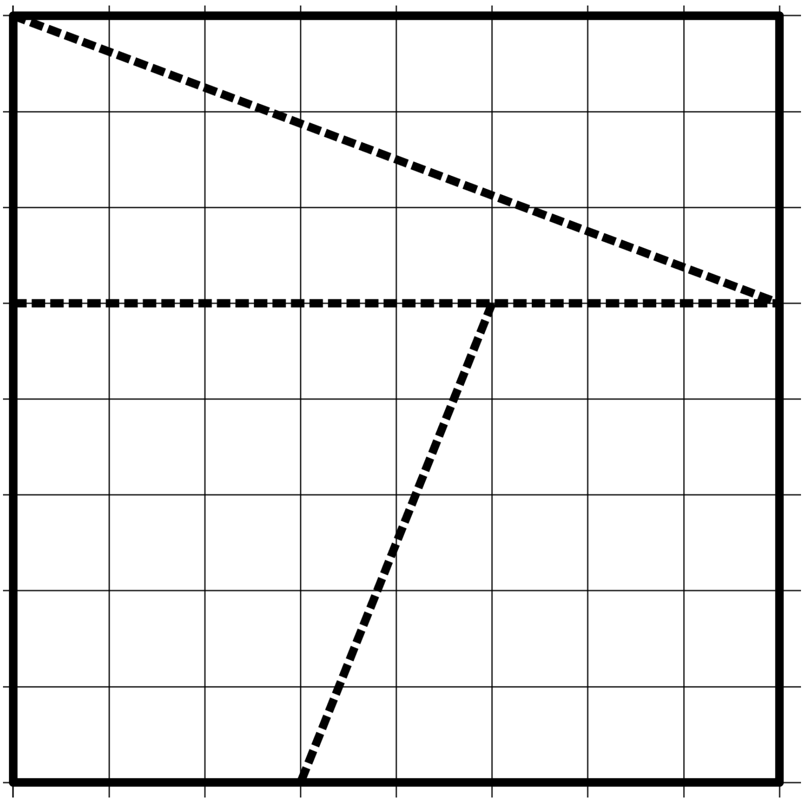
Activity Sheet (II)

1) Find the area of the first square. Then cut the square and make four parts of the square by cutting across the three dashed lines. Make a rectangle using all the four pieces such that you get a rectangle which is NOT a square. Find the area of this rectangle.



Square 1

2) Find the area of the second square. Then cut the square and make four parts of the square by cutting across the three dashed lines. Make a rectangle using all the four pieces such that you get a rectangle which is NOT a square. Find the area of this rectangle.



Square 2