Counting Coins, Bees and Dominoes

Introduction

Let us start by solving some problems.

Part 1: Let us count!

Task 1.1:
Solve the problems given below:

1. A ticket-vending machine accepts only Re. 1 and Rs. 2 coins. It gives out tickets of different values.

For example, if you want a Re 1 ticket then you can put the coins only in one way, namely
Step 1: Put a 1 Re coin.
But there can be different ways of getting a ticket of the same value, like if you want a Rs 2 ticket then you have two ways of getting it.
Way 1. *Step 1*: Put a 1 Re coin, *Step 2*: Put another 1 Re coin
Way 2. *Step 1*: Put a Rs 2 coin
These ways can be written as Way 1: (1, 1), and Way 2: (2).
In the case when you want a Rs 3 ticket, what are the different ways you can get it?
Way 1. *Step 1*: Put a Re 1 coin, *Step 2*: Put a Re 1 coin, *Step 3*: Put a Re 1 coin, thus (1, 1, 1).
Way 2. *Step 1*: Put a Rs 2 coin, *Step 2*: Put a Re 1 coin, thus (2, 1).
Way 3. *Step 1*: Put a Re 1 coin, *Step 2*: Put a Rs 2 coin, thus (1, 2).

Are there any more ways you can get a Rs 3 ticket?

Fill the entries in the table with various ways in which you can get tickets with different values.

Cost of ticket	Ways	Number of ways
1	(1)	1
2	(1, 1), (2)	2
3	(1,1,1), (2,1), (1,2),	3
4		
5		
6		

Table 1

2. All of us must have seen a honeybee hive. Honeybees make those hives to store nectar or honey. In a honeybee hive there are three types of bees; the queen bee, worker bees and the drone.

The queen is a female bee who produces eggs. The worker bees are also females but they do not produce any eggs. The drones are the male bees who do not do any work.

All female bees have 2 parents, a male and a female. And the drones have only one parent, a female bee.

Here we do not follow the convention of Family Trees that *parents appear above their children.* In this family tree, the latest generation is at the top and the lower we go, the older the bees get. In this tree all the *ancestors* of the bee are below the bee.

The family tree of the male bee will look something like this:

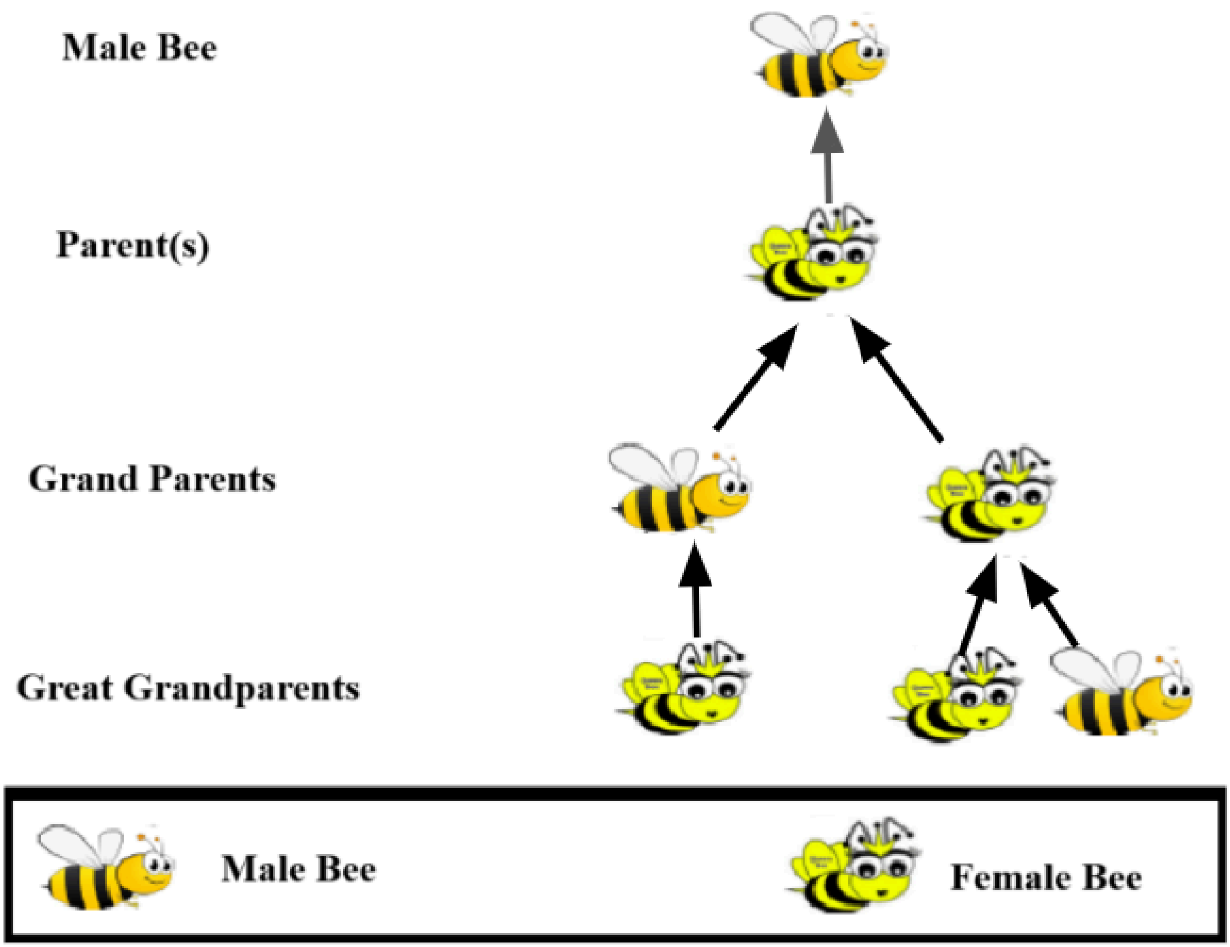


Figure 1: Family Tree of a male bee

Extend the family tree of male bees and draw one for female bees. And then complete the table given below.

No.	Parent	Grandparents	Great-Grand-Parents	GT-Great-Grand-Parents	GT-GT-GT-Grand-Parents	GT-GT-GT-GT-Grand-Parents	GT--GT-GT-GT-GT-Grand-Parents
Of male bee	1	2	3				
Of female bee	2	3					

Table 2

3. Suppose we have unit squares arranged in a $2 \times n$ grid. See the figures below for examples.

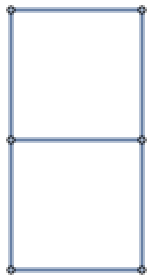


Figure 2: 2×1 grid

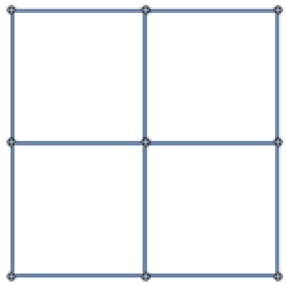


Figure 3: 2×2 grid

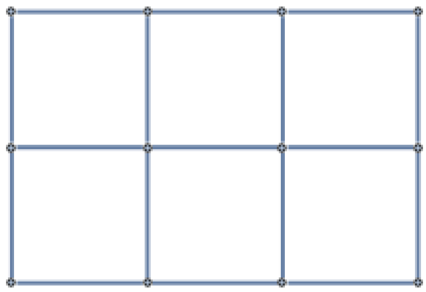


Figure 4: 2×3 grid

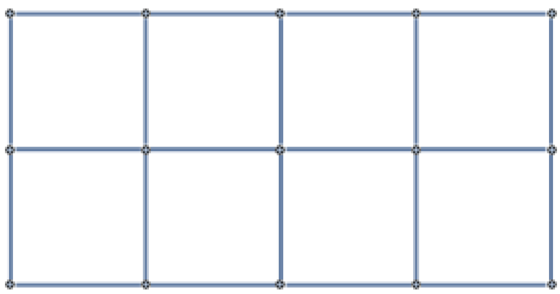


Figure 5: 2×4 grid

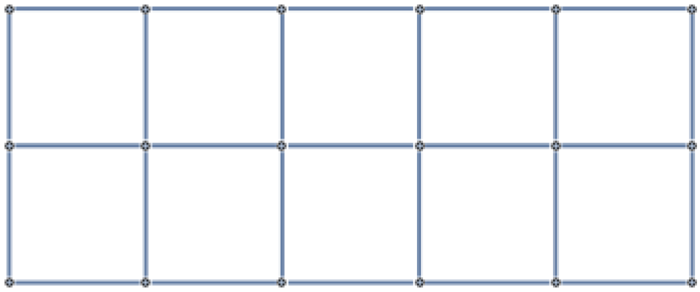


Figure 6: 2×5 grid

A **domino** is a collection of two adjacent squares. There are two possible dominoes, **vertical** or **horizontal**, as shown below.

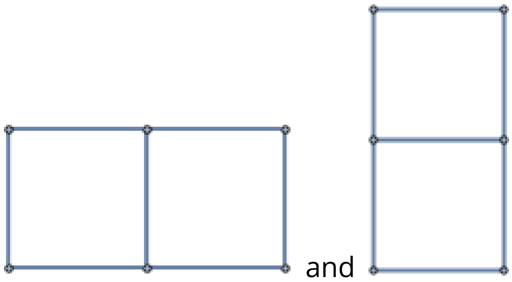


Figure 7: Two possible dominoes

How many ways are there of completely filling this $2 \times n$ grid with no overlaps. Use the dominoes given in the Activity Sheet (I) at the end of this unit. Cut the dominoes given on the sheet and find all the ways of tiling them on the rectangles in Figure 1, 2, 3 and 4 and fill the table given below.

Value of n	1	2	3	4	5	6
Number of tilings						

Table 3

After you have solved all three questions, discuss your answers with your friends. Did you solve all the problems till the end or you found a pattern and predicted your answer from it? How can you be sure that your prediction is correct?

We saw that these numbers come in different situations. Maybe that's why mathematicians find them interesting.

The sequence of numbers you got while solving the three problems in the previous part are called **Fibonacci Numbers or Fibonacci Sequence.** These numbers appear unexpectedly in mathematics, so much so that there is an entire journal dedicated to their study, the *Fibonacci Quarterly*.

History

The oldest known appearance of the Fibonacci numbers is in India. It is known that Pingala (roughly 450–200 BCE), the ancient Indian mathematician and poet, studied them in the context of Sanskrit poetry. Later, Virahanka (c. 700 CE) wrote a detailed study about these numbers. Though a lot of Virahanka's work is lost, one can find references of his work in the later work by Gopala (c. 1135 CE).

These numbers are named after the mathematician, Leonardo Fibonacci (son of Bonacci), also known as Leonardo of Pisa. Fibonacci talked about these numbers in his book, *Liber Abaci*, in the growth of populations of rabbits, which was published in 1202. Do try to find out about these mathematicians from the internet or your library.

Part 2: Fibonacci Spiral and Golden Ratio

Task 2.1: Constructing the Fibonacci Spiral

Another term that is often mentioned in association to Fibonacci numbers is the '*Fibonacci Spiral*'.

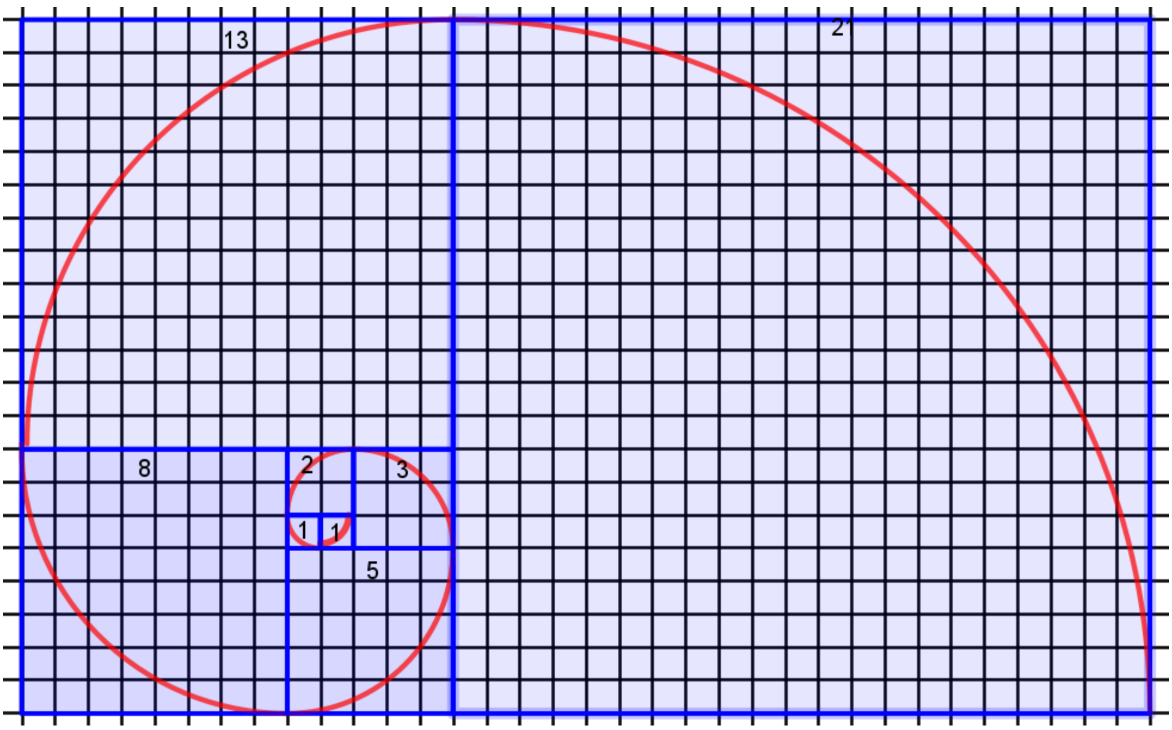


Figure 8: Fibonacci Spiral

Let us construct this spiral and see why it is associated with the Fibonacci numbers.
Start with a square of side 1 unit, in the middle of the grid paper provided to you as shown in Figure 9.

1st Fibonacci Number = 1

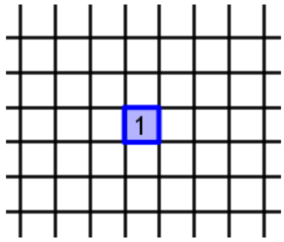


Figure 9

Then on the left of this square, draw another square of side 1 unit as shown in Figure 10.

2nd Fibonacci Number = 1

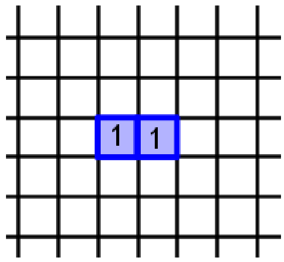


Figure 10

Now on top of these two squares draw a square of side 2 units such that it shares a side with both the above squares as shown in Figure 11.

3rd Fibonacci Number = 2

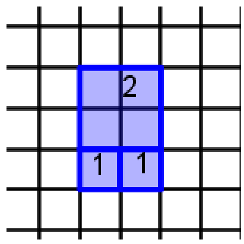


Figure 11

Now the combined figure is a rectangle. What is the height of this rectangle? ____________ units
What is its breadth? ________________ units

On the right of and adjoining this rectangle, draw a square whose side is equal to the height of this rectangle. Now you get another rectangle. What is the height of this rectangle? ____________ units
What is its breadth? ________________ units

Now draw another square below the new rectangle such that its side is equal to the longer side of the rectangle.

Continue this process, left, top, right and bottom till you get a rectangle with one of the sides as 34 units.

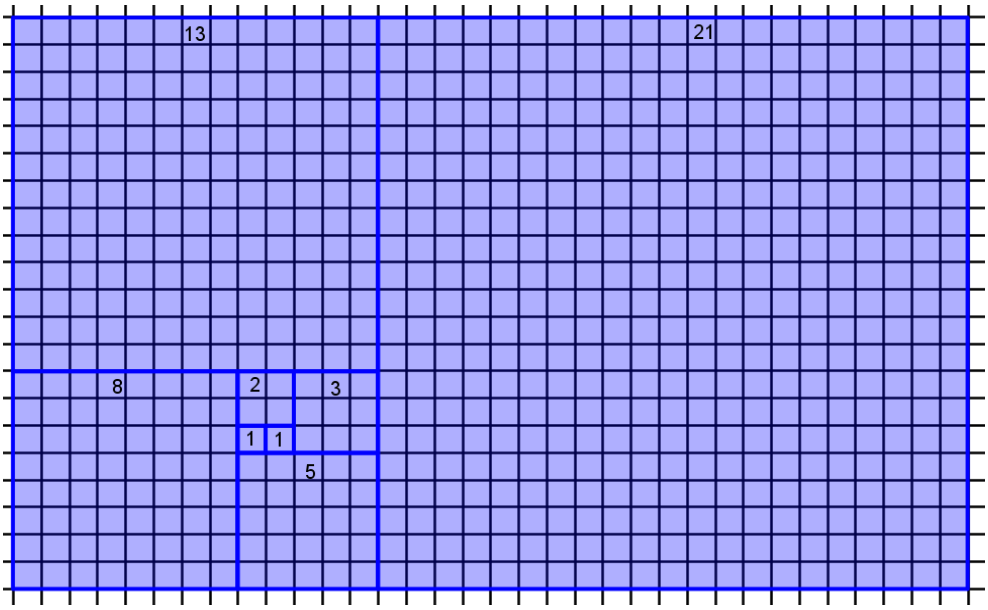


Figure 12

Now draw a quarter circle in the first square, it will look like this.

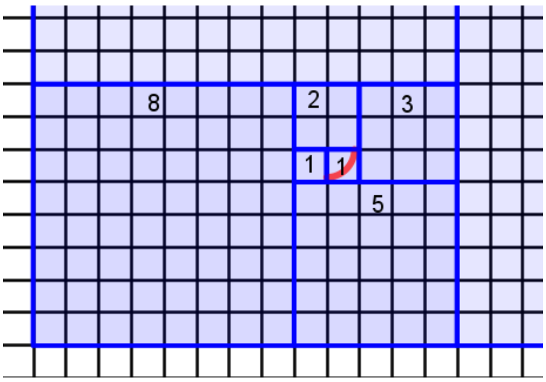


Figure 13

Continue the above arc and draw another quarter circle in the next unit square to get a semicircle.

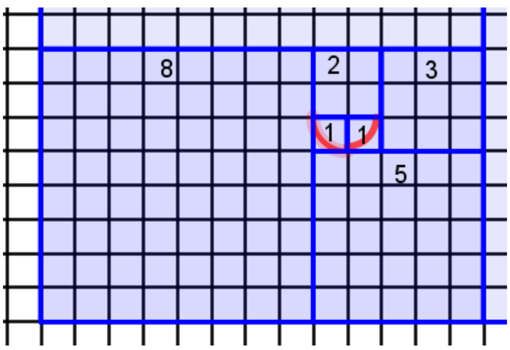


Figure 14

Continue this process of drawing quarter circles in the same direction as you drew the squares. Finally, you should get a fibonacci spiral similar to Figure 8.

This is called the **Fibonacci Spiral**. Spirals very similar to the Fibonacci Spiral occur widely in nature in various settings, e.g., shapes of galaxies, cyclones and snail shells. Try to get more information about this spiral from your library or the internet.

Figure 15

Apart from the fact that they can be seen in nature, there are other reasons why mathematicians find these numbers interesting. In the next part of this unit let us see why.

Task 2.2: Finding the Golden Ratio

Find the ratio between two consecutive Fibonacci numbers. You can use a spreadsheet or an Excel sheet for this. Notice what happens to the ratio as n increases. Your table should look like this.

n	F_n	$\frac{F_n}{F_{n-1}}$
1	1	
2	1	1.000
3	2	2.000
4	3	1.500
5	5	1.667
6	8	1.600
7	13	1.625
8	21	1.615
9	34	1.619
10	55	1.618
11	89	1.618
12	144	1.618
13	233	1.618
14	377	1.618
15	610	1.618
16	987	1.618
17	1597	1.618
18	2584	1.618
19	4181	1.618
20	6765	1.618

Table 4

Part 3: Patterns of Fibonacci numbers

Task 3.1: For different values of n, add the first n Fibonacci numbers. Does it look close to another Fibonacci number?

Task 3.2: Missing Area?

In the Activity sheet (II) given at the end of the unit, you will find two squares. Find the area of the square no. 1 and then cut it (on the dotted lines) and arrange the four pieces to form a rectangle (which is not a square). What is the area of the new rectangle?

Similarly, first find the area of square no. 2 and then cut it (on the dotted line) and arrange the four pieces to form a rectangle (which is not a square). What is the area of this new rectangle?

Observe the lengths of the squares given to you and the lengths of the rectangles you got. Have you seen these numbers before?

Let us see why this happens:

Complete the following table:

No	Fibonacci number	Square of the Fibonacci number	Product of previous number and next number
n	F_n	F_n^2	$F_{n-1} \times F_{n+1}$
1	1	--	--
2	1	$1^2 = 1$	1 x 2 = 2
3	2	$2^2 = 4$	1 x 3 =
4	3		
5	5		
6	8		
7	13		
8	21		

Table 5

Can you see any pattern emerging from the table above?

Task 3.3: More Patterns in the Fibonacci Sequence

1. Look at the numbers given below and see if you can get many more patterns from these numbers. You can also extend the pattern if you want.

1, 1, 2, 3, 5, 8, 13, 21, 34,

2. If we write the Fibonacci sequence as follows:

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	...
F_n	1	1	2	3	5	8	13	21	34	55	89	144	233	377	610	...

Table 6

$F_3 = 2$. Now, what can you say about every 3rd number in this series, $(F_6, F_9, F_{12}, \ldots)$
$F_4 = 3$. Now, what can you say about every 4th number in this series, $(F_8, F_{12}, F_{16}, \ldots)$
$F_5 = 5$. Now, what can you say about every 5th number in this series, $(F_{10}, F_{15}, F_{20}, \ldots)$

Do you see any patterns from this table?

Extend the table and check if your patterns for other Fibonacci numbers too. You can also use a spreadsheet or an Excel sheet to do this.

3. If we square the numbers in Fibonacci series and make a new series by adding the consecutive squares, what do we get?

Observe the table carefully and discuss the pattern you got with your classmates.

F_n	1	1	2	3	5	8	13	21
F_n^2	1	1	4	9	25	64	169	441
$F_n^2+F_{n+1}^2$	2	5	13					

Table 7

4. If we add the squares of the first n Fibonacci numbers, what do we get?
Observe the table carefully and discuss the pattern you got with your classmates.

n	1	2	3	4	5	6	7	8
F_n	1	1	2	3	5	8	13	21
F_{n+1}	1	2	3	5	8	13	21	34

F_n^2	1	1	4	9	25	64	169	441
$F_1^2 + \cdots + F_n^2$	1	2	6	15	40	104	273	

Table 8

Extended Task

Part 4: A formula for Fibonacci Numbers

We see that Fibonacci numbers are very important. But the recursive way to get them is very tedious. To get the 100th Fibonacci number it would be necessary to find the 99th and 98th numbers. Wouldn't it be nice to have a simple form for all Fibonacci numbers?

Look at the expression given below:

$$F_n = \frac{\left(\frac{(1+\sqrt{5})}{2}\right)^n - \left(\frac{(1-\sqrt{5})}{2}\right)^n}{\sqrt{5}}$$

Substitute $n = 1, 2, \ldots$ and check. What did you get? Do you think this will give us a Fibonacci number? Use a spreadsheet and check what numbers you get for different values of n.

So, now we know how to find Fibonacci numbers. Try to find more interesting patterns in these numbers.

Part 5: Taj Mahal and it's Golden Rectangles

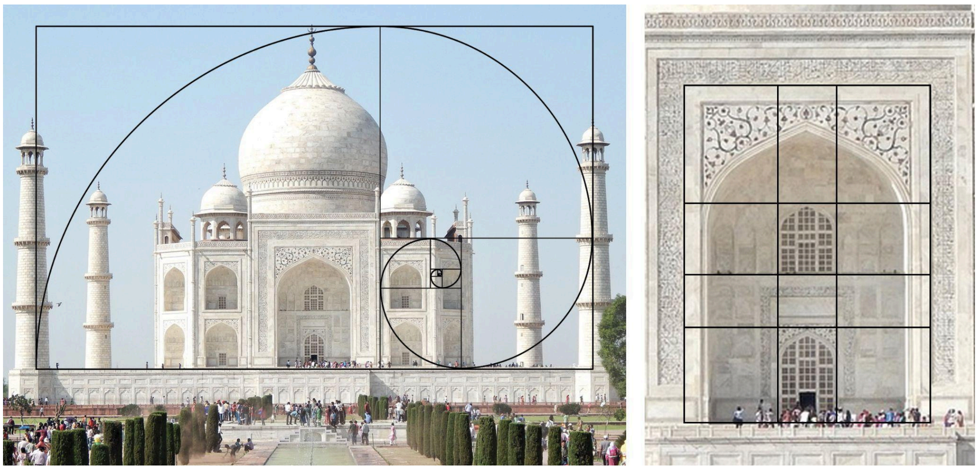


Figure 16

A rectangle whose side lengths are in the golden ratio, that is, 1:1.618 is called a golden rectangle. One characteristic that distinguishes this shape apart is that when a square chunk is taken out, what's left over is another golden rectangle, meaning it has the same proportions as the initial one.

Indian architecture has long made extensive use of the Fibonacci sequence. The Taj Mahal, one of the seven wonders of the world, follows this pattern. The Taj Mahal is not a perfect golden rectangle, but its overall proportions are close to the golden ratio.

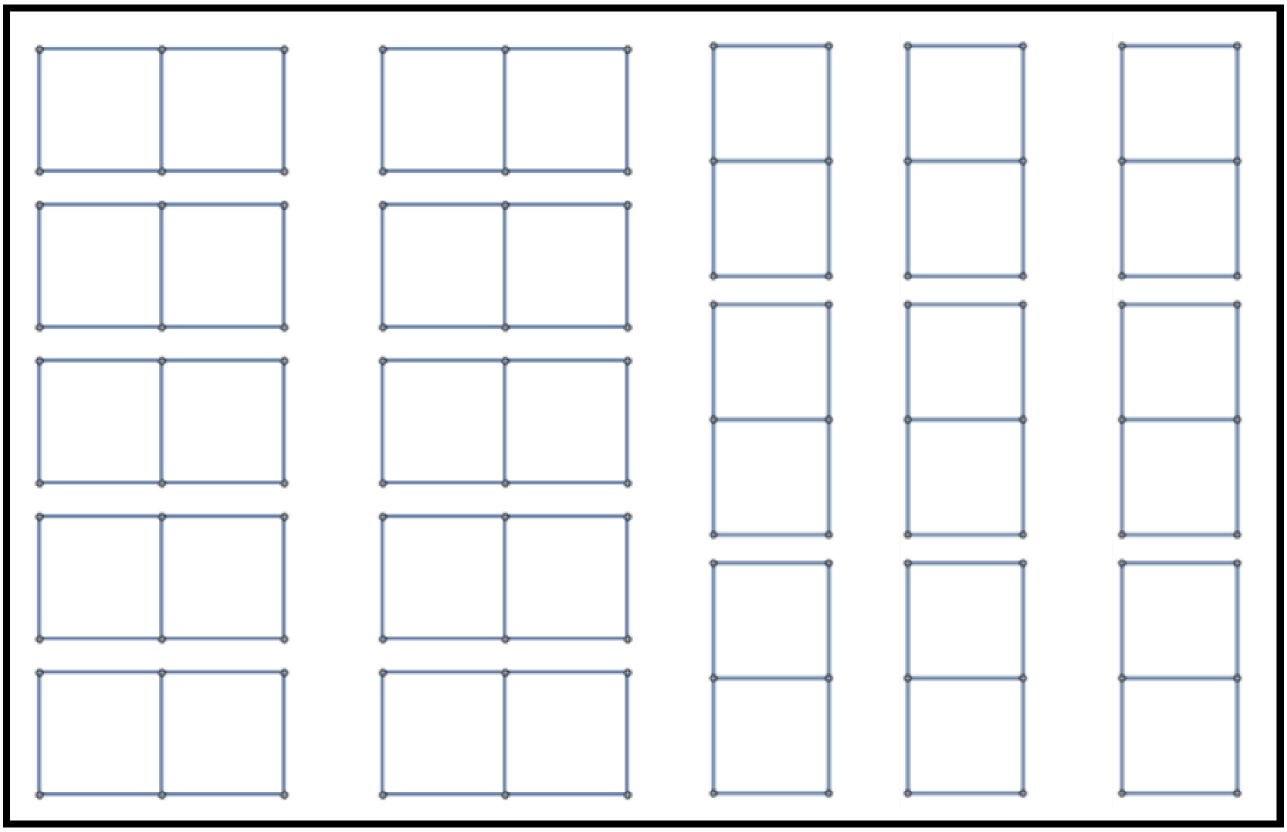
Are there any other golden shapes associated with the Fibonacci spiral or golden ratio?

1.Suggested resources

2. Wikipedia article on the Fibonacci sequence, https://en.wikipedia.org/wiki/Fibonacci_sequence
3. Youtube video on "Fibonacci sequence in nature", https://youtu.be/nt2OlMAJj6o
4. Youtube video on "Nature by Numbers | The Golden Ratio and Fibonacci Numbers", https://youtu.be/me6Dnl2DOtM

Activity Sheet (I)

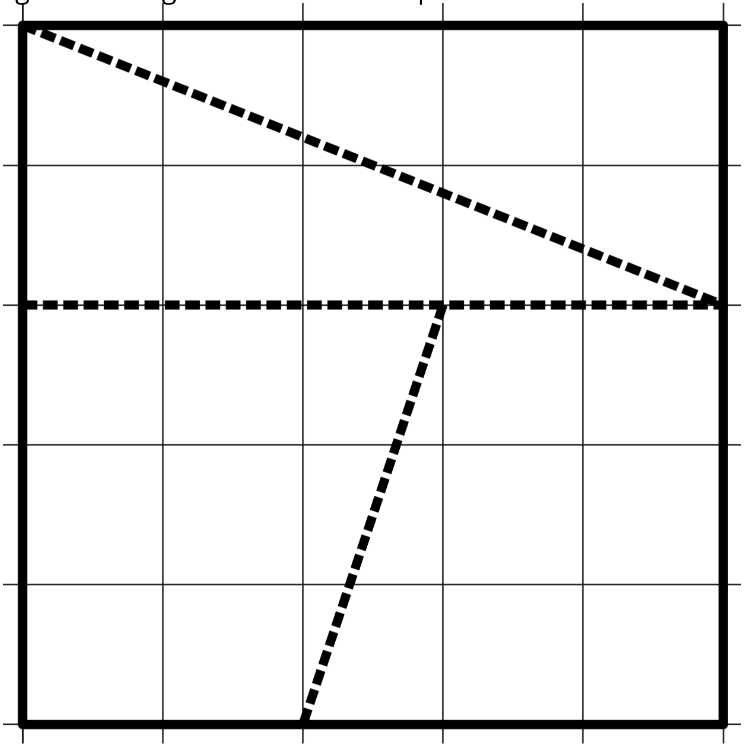
Cut dominoes from the column on the right to tile the grids on the left



Blank page

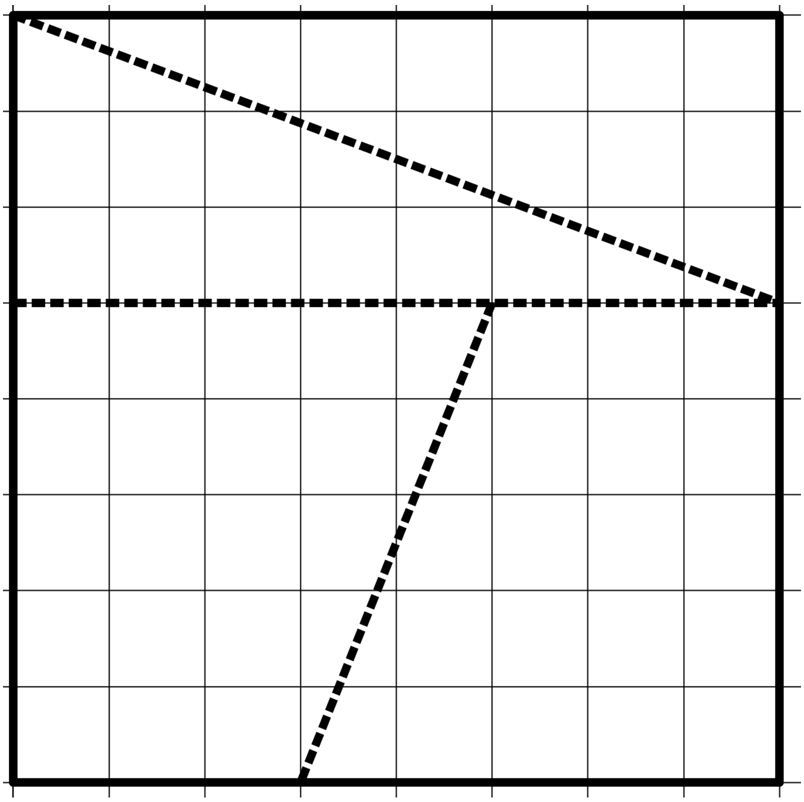
Activity Sheet (II)

1) Find the area of the first square. Then cut the square and make four parts of the square by cutting across the three dashed lines. Make a rectangle using all the four pieces such that you get a rectangle which is NOT a square. Find the area of this rectangle.



Square 1

2) Find the area of the second square. Then cut the square and make four parts of the square by cutting across the three dashed lines. Make a rectangle using all the four pieces such that you get a rectangle which is NOT a square. Find the area of this rectangle.



Square 2