'Music': What it means?

Overview

In this Learning Unit, we will explore a mathematical relation very centralto music, through patterns in the frequency produced by different keys of harmonium/synthesizer. Once we are able to quantify this relation between different musical सुर produced by harmonium/synthesizer, we will proceed to understand the working of Jaltarang, after which we will make one Jaltarang of our own.

This Learning Unit is best conducted with the help of music teacher in the school, who can help students better understand concepts related to music.

Science or mathematics pre-requisites:

- Sound as waves
- Basic idea of frequencies
- Ratios and Proportions

Materials required:

- A mobile phone (could be teacher's) with 'Aurdino Science Journal' app (or any other app with similar functions) installed. (Search for 'Aurdino Science Journal' on Google Play Store)
- A harmonium/an electronic synthesiser
- Ceramic/metallic/glass bowls or beakers, measuring cylinder
- A pencil and a glass stirrer / rod
- Water

Type of Learning Unit: Classroom/Musicroom/Laboratory/ any place which does not have a lot of background noise

Minimum time required: 3 sessions of 40 minutes each.

Links to ciricullum:

- NCERT Class VI Mathematics textbook: Chapter 12 – Ratio and Proportion
- NCERT Class VII Mathematics textbook: Chapter 2 – Decimals and Fractions
- NCERT Class VIII Science textbook: Chapter 13 – Sound

Unit-specific objectives:

- To understand 'musical notes' in a quantitative way.
- To measure sound frequencies using a smart phone application.
- To observe patterns in numbers (or relationship among these) and to articulate these patterns clearly.
- To make use of mathematical patterns in creating musical octaves with different base frequencies.
- To understand the working of Jaltarang and how to make it.
- To understand that the sound (it's frequency) produced by a container changes if the amount of liquid (height of air column) in the container is changed.

Some useful musical terminologies: (These are not essential, but are very helpful to understand the unit better)

- **Volume** – The degree of loudness or intensity/amplitude of sound.
- **Note (स्वर)** – Frequency of a particular सुर.
- **Octave** - an interval of notes, where the frequency of the highest note of that interval is twice that of the lowest note.
- **Pitch** – In Physics, we sometimes use 'pitch' and 'frequency' interchangebly. In music, the reference frequency to the octave is called as pitch.
- **Tempo (लय)** – How slow or fast is the music.
- **Scale (पट्टी)** – In music, 'scale' and 'pitch' are used interchangebly. They both imply the reference सा of an octave (and not the entire octave).

Introduction

Read following conversation between Rahi and her music teacher.

Teacher: For the annual programme, I need one person from your class to sing national anthem from the stage.

Rahi: All of us like our national anthem. So you can select anyone of us.

Teacher: True. But I will take an audition and select the student who can sing it 'properly'.

Rahi: What you mean by properly? All of us know the exact words of anthem.

Teacher: Yes. But you don't have to just 'recite' it, you should be able to 'sing' it.

Rahi: But all of us also know the tune of the national anthem. So all of us can sing it.

Teacher: Singing is not just knowing the tune. You should sing with exact 'सुर'.

Rahi: I know the seven सुर(Sur) . They are सा(Sa) , रे(Re), ग(Ga), म(M), प(P), ध(Dh), नी(Ni).

Teacher: Correct. Singing properly means each sound from your vocal chord should hit correct position of the respective सुर.

Rahi: Correct position? What is that? Are they standing somewhere?

- A typical harmonium has 3 octaves – lower octave (मंद्र सप्तक), middle octave (मध्य सप्तक) and higher octave (तार सप्तक). The middle octave is the natural octave of the instrument.
- In an octave, 7 notes (सुर - सा to नी) are called Pure notes (शुद्ध सुर) . Some notes also have a softer (कोमल) and harsher (तीव्र) version. Look at the image to the right,the pure notes सुर (encircled in red) on the left column typically correspond to the white keys.
- The additional सुर on the right then correspond to the **black keys** on a harmoniuman/synthesiser**.** (Note that this is true only if the reference सा is White 1 key, else this does not hold true.)
- Notice the positions of the additional सुर. The softer notes (represented by *) come before their pure version (as in the case of रे, ग, ध and नी, while the <u>harsher</u> note (represented by #) come after its pure version (as in the case of म). These terminologies used (harsher and softer) are a matter of convention. For eg, the sur between सा and रे is called softer रे, but one could have called it as harsher सा too.

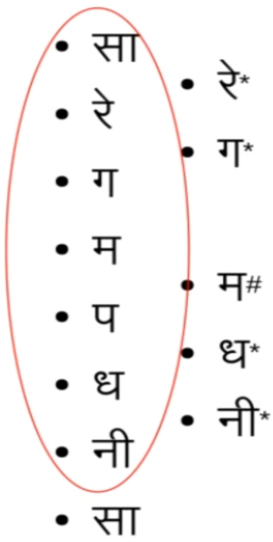

Illustration T1: *सुर in a harmonium*

Q1. What do you think the teacher is trying to say? Do you understand what she means by 'correct position'?

> Play some keys on the harmonium/synthesizer. Preferably, select 2-3 keys which are well separated. After discussion, students may conclude that different keys were played or may say that 'pitch' of the sound was different.
>
> Next, play the same key but with differing amplitude (volume). This is easy to do on electronic synthesiser. But experienced music teacher can also do it with regular harmonium. Let students discuss what was different in the two cases.
>
> Also, play the 7 Sur in 2 different octaves and ask what is the difference between the two. As the octave changes, their frequency also changes.

Q2. Let us take an example of a harmonium. You heard it just now. What did you notice?

Q3. Do you know other examples or instruments where you can produce different sounds?

> Students may give examples of different musical instruments like guitar, flute, wind chimes, jal tarang etc. In each case, ask the respective student to describe the instrument for the benefit of other classmates. The music teacher can provide more examples of musical instruments and their mechanisms of producing different sounds.

You know from the science textbooks that sound is carried to us in form of some kind of waves.

Q4. When we say a uniformly moving 'wave' what all quantities would we need to describe it?

> Through discussion students may think about amplitude and separation between maximas (wavelength). In some cases, they already know relation between sound speed, wavelength and frequency for a uniform wave. Some students may also talk about sound waves with multiple frequencies, but our focus is on notes with single frequency. If not, guide the discussion towards introduction of the term 'frequency (or frequencies) of the sound'. Once the entire group is comfortable with the idea of frequency, proceed to next task.

For the following tasks, we will need a smartphone. There are many smartphone apps which show you frequency of the sound played in the vicinity of the phone. We will use one of those apps.

> We have tested this task with two apps 'Aurdino Science Journal' and 'DaTuner Lite'. Any frequency measuring or musical tuning app or instrument would work too, as long as it gives correct readings for all frequencies between 50Hz to 4000Hz, which is the frequency range we will explore in this LU. If teachers are using another app, they should test it for this frequency range before conducting the LU.

Note: The 'Aurdino Science Journal' app can record and save your data which DaTuner app (and many other musical tuning apps) isn't able to. Also, Audrino Science Journal can be used for various other science experiments.

Task 0 – Familiarization with the 'Aurdino Science Journal' app

We will be using 'Aurdino Science Journal' app throughout this LU to measure the frequency of different musical notes.

Aurdino Science Journal

Here's how to use the app:
1. Download from Play store (it's a free app)
2. Open the app. Click on the '+' (plus) sign on the bottom right, and it'll start a new experiment.
3. This will open an 'Untitled experiment' page. Use the pencil icon on top right to name your experiment and add description if you want.
4. Next you'll have to chose the sensor which will be used for the experiment. This can be done by clicking on the 'Sensors' button from the toolbar at the bottom of the screen.
5. Now you'll see the list of all sensors available on your phone. Choose the 3rd sensor which has the icon of a musical note. This will allow you to measure the frequency of the sound produced.
6. Now you are ready to begin the experiment. You'd be seeing a live graph with frequency on the y-axis and time on the x-axis. You can also look at the exact values of the frequency produced on the top right of the graph.
Note: This app is quite sensitive and it will pick up background noise as well.

Just to test the app, we will need some volunteers. Use a smartphone with permission of your teacher. One of you can try producing the sound 'AAA'(Hindi vowel 'आ') in an extended way and see what frequency gets displayed. Tell the frequency to the entire class and the pass the phone to next group.

You can also use the 'Da Tuner' app. The big alphabet displayed in the centre corresponds to the western musical note associated with the frequency which is displayed at the lower left corner. If the students ask question about the alphabet, briefly explain that it is as per western musical notes.

Da Tuner App

Different discussions can happen at this stage. First and foremost, if the students are not familar with the units 'Hz' (Hertz) and 'dB' (decibels), there should be some discussion to introduce these units.

Another easy observation would be the natural frequency of voice of girls (~210Hz) and boys (~130Hz) will be different. There can be discussion on the point that female voice naturally has a higher pitch than male voice. Some students may try several different notes themselves. Again there can be discussion on what did he/she exactly change when producing different notes. Which frequency was higher, which was lower etc.

Task 1 – Understanding the relation between different notes on a harmonium

Teacher will play different keys on Harmonium and you have to note frequency of those keys in the table below. For convenience, let us agree to a convention. On the harmonium, you will see a pair of black keys and then a set of three black keys. The white key just before the black pair (first key in the figure) will be called White 1 (W1).

Illustration 1: *Arrangement of black and white keys in a harmonium.*

As you proceed rightwards from this key, next key will be called Black 1 (B1), the next one is W2 and so on. Note that B3 comes after W4.

Note: Since the app shows you live data, it might be difficult to note down the relevant frequencies very quickly. Use the 'record' button to record the relevant part, save it and then use it later.

Once you are done playing the different सुर and done with the recording, open the recording and it'll show you the graph of the entire recording. You can zoom in and find out the frequency of the various सुर you played.

While noting down frequencies for each सुर, you may note only one digit after the decimal point.

> Any musical instrument doesn't hold the exact same frequency for long, thus some variation of a few hertz is expected. Students should not worry about it.

Table 1: Frequency Table

Key	Freq.	Key	Freq.	Key	Freq.
W1		W8		W15	
B1		B6		B11	
W2		W9		W16	
B2		B7		B12	
W3		W10		W17	
W4		W11		W18	
B3		B8		B13	
W5		W12		W19	
B4		B9		B14	
W6		W13		W20	
B5		B10		B15	
W7		W14		W21	

Q5. Do you see any patterns or relationships among numbers in this frequency table? List them down and discuss with the class.

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> Students may give some of the below listed responses, or some altogether new responses. It is important for the teacher to acknowledge the pattern, discuss it with the class and encourage students to check if it is true for all frequencies, generalize it or find exceptions to it depending on the pattern.
>
> - The frequency is increasing
> - Constant difference in the lower keys (Not true for all keys)
>
> Eventually these patterns should emerge out: (Give students enough time to reach to these patterns)
>
> - The frequencies are almost doubling in all rows (Horizontally)
> - Difference of consecutive notes increases (Vertically)
>
> If students are not able to come up with these on their own, the teacher can gradually lead the discussion so that students see these patterns.

Try finding out the ratios of 2 consecutive frequencies, for eg. B1/W1, W2/B1and so on.

> Students may make a new table with a column for the the ratio of consecutive frequencies. This ratio would be very close to 1.06 every time.

Typical value of the ratio (r) is = ________________

> So far we have seen,
>
> - Moving horizontally in the same row, the frequency nearly doubles.
> - Ratio of consecutive keys is same.
>
> Can we use these 2 observations to mathematically predict the ratios?
>
> Let f is the first frequency, r be the common ratio, then the second frequency will be f × r, third frequency will be f × r × r, i.e., $f \times r^2$. Similarly each of the later frequencies can be found out by multiplying the common ratio 'r' with the previous frequency.
>
> So if W1 is f, then W8 (12^{th} key from W1) will be? - $f \times r^{12}$
>
> But we also know that W8/W1 is 2, using this can you find out the value of r?
>
> r= $\sqrt[12]{2}$, = 1.059)

Let our first frequency (White 1) be f. Then second frequency (Black 1) will be f × r. Third freqeuncy (White 2) will be $(f \times r) \times r = f \times r^2$.

In this notation, freqency of White 8 will be $f \times r^{\_\_}$.

Find ratio of the frequencies of White 1 and White 8. Ratio = __________________

Hence value of r can be expressed as a power of 2 as __________________

> This common ratio is a feature of Harmonium-like instruments, which follow equi-tempered scale, where consecutive keys have the same ratios.

Task 2 – Understanding the seven सुर for any given scale

In the beginning, Rahi spoke about seven सुर in Indian music. Following table gives a relation between 'सुर' and harmonium keys (in a particular scale). Refer to the table above and note down the frequencies of these keys. Now find ratio of each frequency in this table with the frequency of the first key (W1). Note down the ratios as a fraction instead of converting them into decimals, as it'll be helpful later.

Table 2: Finding ratios

सुर	सा	रे	ग	म	प	ध	नी	सा
Key	W1	W2	W3	W4	W5	W6	W7	W8
Freq.								
Ratio	1							

> Some students may instead choose keys from W8 – W15 and some can take W15 - W22. They will realise in case of each octave, they are getting almost same ratios. They can all share their ratios and get mean ratio for each सुर.

Harmonium or piano uses pre-defined frequencies which are set to a fixed frequency ratio. This is called 'equi-tempered scale'. However, one may note that there are other ways of defining scale, which give almost same frequencies. Looking at the ratios obtained from the table above, one may notice that these ratios can also be expressed as fractions where both numerator and denominator are both integers less than 20. Write the ratios in that form.The sequence of ratio you get is known as 'Ptolemaic Sequence'.

Task 3 – Finding the frequency of seven सुर in any scale

Different scales in piano or harmonium just mean starting your first सुर at another key. Now suppose your first सुर (i.e. सा) is starting with B1 instead of W1. Use the ratios you found above and the table on the previous page, decide which keys will correspond to other सुर. Here B1 is taken just as an example, you may choose any other key instead of B1.

Table 3: Finding a different scale

सुर	सा	रे	ग	म	प	ध	नी	सा
Ratio								
Freq.								
Key	B1							

Play this sequence on harmonium to see if you get similar sequence of sounds as playing W1-W7.

> Some students may instead choose some other keys, apart from from W8 and W15, as their first सुर and try playing them on the harmonium.

Try to represent these ratios that you got by simple fractions, where the numerator and denominator are fairly small integers.

Usually when converting decimals to fractions, it is useful to look out for certain decimals: 0.125 = 1/8, 0.25 = 1/4, 0.75 = 3/4 , 0.33 = 1/3, 0.66 = 2/3, 0.5 =1/5 and so on. For eg. if one of the ratios is 1.12, then 0.12 is close to 0.125. So it can be fractionalized as 1 + 1/8 = 9/8.

Suppose the ratio of W19 and W15, i.e the प key with the base सा is 1.49. This could be taken as 1.50 to obtain a simple fraction of 3/2. Take this opportunity to discuss with students why this is a valid approximation. The reasons include – We have approximated the frequencies while recording, the musical instrument (harmonium in this case) might not be perfectly tuned, there might be slight disturbances while recording the frequencies, we have truncated the ratios of frequencies to 2 decimal places. All these factors may lead to the final ratio of प and base सा to be 1.49 instead of 1.50. So to obtain a simple ratio, we can take it as 1.50 and get 3/2.

And what we are looking for here is simple fraction, which can be very close to the ratios we got, but not exactly the same.

रे and नी are slightly non-trivial and students may take some time, or need some hinting to arrive at their fractional value. The fractions for other सुर can be arrived at easily.

Here is the table for the सुर and their simple fractional ratios:

सुर	(base) सा	रे	ग	म	प	ध	नी	(upper) सा
Ratio	-	9/8	5/4	4/3	3/2	5/3	15/8	2

Possible extension: Some students may like to explore the idea of vocal frequencies further. They may like to match their सुर with that of harmonium. They may try singing a song and see response on the app. Teachers may note that generally vocal frequencies don't exactly match with the frequency of harmonium. Even reasonably good singers can be off by 1-2Hz. For bad singers (ones who we call बेसुरे), they can be off by a larger difference (5-10Hz). Most listeners would not pick up small differences in vocal and instrumental frequencies, but trained singers and good listeners have trained their ears to be more sensitive than most other humans.

Human ear is trained to appreciate particular multiples of base frequency, what we call as harmonics. So when this happens, it's literally 'music to our ears' but when the frequencies played/sung are not these simple fraction multiples of the base frequency, it feels that the sound is out of tune.

Task 4 – Understanding Jaltarang

Have you heard the sound made by water while filling up a glass or a water bottle? The sound made by the vessel keeps changing as it gets filled. This tells us that we can adjust the frequency heard from a vessel by changing the amount of water level inside it.

Jaltarang literally means 'waves in water'. It is a traditional musical instrument used in Indian classical music. You might have seen Jaltarang being played by a musician. It is played by striking a set of bowls which have varied amount of water in them, producing melodious music. Let us understand how it works and then make Jaltarang of our own.

Take ceramic/metallic/glass vessels of different kinds (bowls/beakers/measuring cylinders/drinking glasses) and a measuring cylinder.

Picture 1: *Setting up for Jaltarang*

Q1. Place the side by side and tap them with a pencil and note down the frequency in each case. Which bowl has the highest frequency?

Table 4

Description of the vessel	Frequency observed

> In this part no water should be added to any bowl / beaker. Discuss what has exactly has changed in these different vessels? Either the radius, the thickness or the material of the vessel might be different. In all these cases you will see clear change in the frequency.

Q2. Repeat the same exercise with a glass stirrer / rod. Is there any difference in the resulting frequency.

Table 5

Description of the vessel	Frequency observed

> Here we are just demonstrating that the material of stick will make only a small change, as a compressible material (pencil) will produce a slightly different sound as compared to a rigid stick (glass stirrer).

Q3. Now take the largest beaker, keep adding a fixed amount to water to it (say 25 ml each time) and note the frequency.

Table 6

Volume of water added	Frequency observed

Q4. What happens to the frequency as the volume of water increases?

> Note that as the volume of water in the vessel increases, the height of the air column in the vessel decreases. From the formula mentioned below, decreasing the height of the air column would lead to increasing frequency, i.e they are inversely proportional. But in reality, adding water leads to a decrease in frequency. This may be because it's not a simple system, and many other factors come into picture in deciding the frequency of sound produced.
>
> In this formula, (see reference no. 4 for more details)
>
> a = thickness of walls of beaker
>
> R = radius of beaker
>
> H = height of air column inside beaker
>
> h = height of water-level inside beaker
>
> (note: For an empty beaker,
>
> H = height of beaker & h = 0)
>
> $$\left(\frac{\nu_0}{\nu_h}\right)^2 \approx 1 + \frac{\alpha}{5}\frac{\rho_l R}{\rho_g a}\left(\frac{h}{H}\right)^4.$$
>
> ν_o = frequency of sound produced by empty beaker
>
> ν_h = frequency of sound produced by beaker with water-level upto height h
>
> ρ_g = density of the material of beaker
>
> ρ_l = density of water
>
> Y = Young's modulous of beaker
>
> α = factor of the order of unity

Q5. Is it possible to change the frequency of this bowl/beaker to match that of the smallest bowl/beaker and at what water level will that occur?

Q6. What all parameters are important in deciding vibrating frequency of the beaker?

The mathematical expression for natural frequency of a beaker is very complex (as seen above). So it may not be inferred from the data collected. However, one can conclude that for higher frequencies you need

- thicker side walls.
- smaller density of the material of the beaker.
- smaller height of air column inside the beaker.
- smaller radius of beaker.
- smaller value of Young's modulus (need not be discussed).

It also depends on the shape of the vessel chosen, i.e keeping all other parameters same, a drinking glass, a cylindrical beaker and a bowl may produce different frequencies.

Task 5 – Making of Jaltarang

Take a number of cups/glasses/bowls/small vessels (not just 7). Try out cups of different shapes, sizes and materials (metal, glass, ceramic). Strike them with a glass rod/stirrer/pen and note down their frequencies. Repeat the same procedure when it's almost full with water.

Table 7: Frequency of vessel when empty and filled

Description of the vessel (shape, material etc.)	Frequency observed (Hz)	
	Empty	Filled with water

The reason why we are asking to collect and find out frequencies of more than 7 vessels is to get a wide range of frequencies, which the students will later adjust by adding or removing water from the vessel. Also, it might so happen that some vessels may not give a one particular frequency, but show huge

> variations in it, in the range of hundreds of Hz. This might happen maybe because of some structural crack or fault in the vessel. It is advised to not use such vessels for the making of Jaltarang.

Now you have a range of frequencies which a particular vessel can produce when struck.

As we have seen in the previous tasks, musical notes are multiples of base frequency. Using the range of frequencies available to you, choose your base सा. Note that your upper सा will have twice the frequency of your base सा, so make your decision wisely. Now using this base सा frequency and the ratios of the सुर you got previously, complete the table below:

Note: It might be helpful to number your vessels as 1,2,3... This will make it easier to identify them if all of them are similar.

Table 8: Setting up the Jaltarang

सुर	Frequency ratio (in fractional form)	Actual frequency after choosing your base सा	Which vessel can be used to play this frequency?	Amount of water in the vessel (Empty/partially full, almost full)
सा (Base)				
रे				
ग				
म				
प				
ध				
नी				
सा (Upper)				

The frequency table for your Jaltarang is ready!

Now using this table, arrange the vessel based on their सुर, and add water to them accordingly. Adjust the water level to get your desired frequency. To make minor adjustments in the frequency, you can use a dropper or a spoon to add or remove water in small quantities.

After you are done checking the frequency for each vessel, your Jaltarang is ready! As mentioned earlier, musical notes are just multiples of base frequencies. Strike the vessels in the musical octave order (base सा to upper सा) and listen to it. Does it 'sound' right?

Enjoy playing melodious music using your self made instrument!

Note: Do not keep Jaltarang for too long because frequencies will change as water evaporates, so you may have to retune it before using.

Further explorations with Jaltarang:

1. Students can also explore how the following changes affect the frequencies of a Jaltarang:

(i) adding salt to water

(ii) changing the temperature of water

(iii) taking any other liquid (such as oil) in place of water.

(iv) by changing the part of vessel where you are striking.

2. When striking with sonorous objects (eg. spoon or metal objects) sometimes overtones may be produced with high sound intensity. As a result, the smart phone sensor may pickup the overtone frequency as the observed frequency instead of the basic frequency. If you observe an unexpected high frequency for a vessel with a certain water level, then check if there is another signal with lower frequency (which should be 1/2 or 1/3 of the observed frequency), which may give you the basic frequency of that system.

Different scales in music:

What students explored in this unit is one kind of musical scale. There are actually many kind of musical scales. What is common to all these scales is the octave. As we defined before, the range where the frequency doubles (i.e., f becomes 2f) is called an octave. For example, 100 Hz to 200 Hz is an octave, as is 248 Hz to 496 Hz.

Now, suppose you have to divide an octave into 7 parts, one possible method is to divide it into equal parts.

Then maybe, we can divide using some formula like all divisions are related by some rational number p/q (higher notes are related to base notes as multiples of the same rational number). Or that the ratio of any two consecutive frequencies is $2^{1/12}$ (same for any two consecutive notes).

In music the formula used defines what is called a scale. Having the same p/q is called a just tempered scale and using $2^{1/12}$ formula is called a equally tempered scale. Broadly speaking, Indian classical music uses the just scale and musical instruments such as the harmonium and piano use the equal tempering. This is why many people who pursue classical music usually do not use harmonium in concerts.

The number of notes within an octave need not be always seven. Many musical forms in the world use different number of divisions for an octave. The following list some of them,

– 3 – Vedic Chants called udAtta, anudAtta and svarita

– 4 – American Indian, Inuit, Maori, African music

– 5 – (pentatonic)- south east Asian- Chinese music

– 6 – (hexatonic) – western folk music

– 7 – common Western and Indian classical music

– 8 – Jazz

– 12 – chromatic scale

– 22 – srutis in Indian classical music

– 24 – Arabic music

(You can find examples of all these musical forms on YouTube).

Both western and Indian classical music (Hindusthani and Carnatic) have 7 major notes and 5 minor notes to an octave. The names of these notes in the two traditions are given below. Among Hindustani and Carnatic traditions, there are slight differences in the labels of notes. For example, fourth note which is known as Komal Gandhar in Hindustani tradition, is called sadharana gandhara (g2) as well as shatrushi rishabha (r3) in Carnatic tradition.

Indian tradition	Western tradition	
s	C	
r1	C# (C sharp)	
r2/g1	D	
r3/g2	D# (D sharp)	
g3	E	
m1	F	
m2	F# (F sharp)	
p	G	
d1	G# (G sharp)	
d2/n1	A	
d3/n2	A# (A sharp)	
n3	B	

The students who have studied any of the classical music forms may try to correlate musical frequencies to the names of notes in their tradition. Teachers may encourage such discussion and help them appreciate the diversity of musical scales and mathematical patterns therein!

Suggested Readings

1. A good article which highlights how mathematics is related to music -
https://www.simplifyingtheory.com/math-in-music/
2. A collection of material (articles, videos, podcasts) on music and mathematics by American Mathematical Society -
http://www.ams.org/publicoutreach/math-and-music
3. A good read on history and current status of Jaltarang -
https://highonscore.com/jal-tarang-one-rarely-heard-instruments/
4. French A. P. (1982) *In Vino Veritas A study of Wineglass Acoustics,* retrived from:
http://dobas-phd.home.amu.edu.pl/docs/Drgania kieliszka.pdf

