FINDING THE RIGHT PATH

Unit-specific objectives:

- To represent a problem mathematically using graphs and the concepts of paths in graphs
- To learn switching across representations of the same problem
- To understand importance of abstraction and how it makes a problem more approachable
- To gain experience in forming conjectures and creating useful and counter examples
- To acquire basic knowledge of graph theory concepts and techniques

Materials required:

- Paper, pencil and eraser.
- The activity goes better if students work collaboratively on the problems. This could be enabled by a large poster with an illustration of the Konigsberg bridges and some stickers/pins that can be used to cover/mark bridges that have been used and cannot be used again.
- If a screen and projector are available, they could replace the poster. An animated version could be projected (for example, [Mathigon]) and students could work on it together.

Prerequisites:

- Odd and even numbers
- Basic acquaintance with maps
- Comfort with using symbols to represent a quantity or object

Minimum time required: 90 minutes

Type of Learning Unit: Classroom

Task 1: Seven Bridges of Konigsberg!

Today we are going to begin with the story of Konigsberg in the 18th century, its geography, bridges, and the question asked by its citizens.

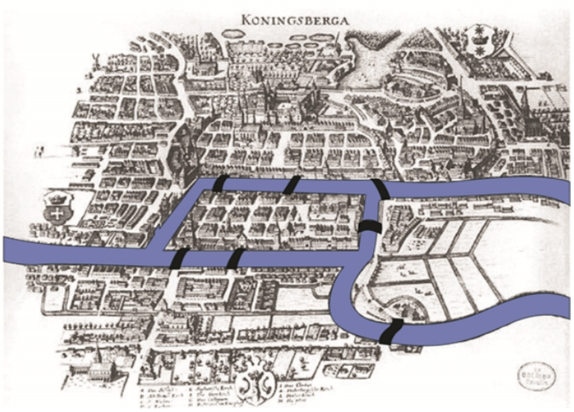

Figure 1

Share a worksheet with an illustration of 18th century Konigsberg. Given below is an example of one such picture. One can also project the same and narrate the story. Make sure the projection is clear and students are able to see all the connections between the river and the land. The story could be told as follows (Source: [BTM])

Kaliningrad is a city which lies between Lithuania and Poland and is at some distance from the rest of Russia. In fact, it was originally a German town and was called Konigsberg. The river that runs through this town was then called the river Pregel. The Pregel branched and looped through Konigsberg, as shown in the picture, and in the eighteenth century there were seven bridges across it.

A challenge took shape around the river and the bridges. Is there a route that would let one walk across all seven bridges exactly once? No bridge could be missed or crossed twice and, of course, there was to be no swimming across the river!

Once the story has been shared, ask students to try to sketch a successful path on the illustration in their worksheets. Give them about 5 minutes to try their hand at this and use this time to walk around the class and make sure everyone has properly understood the problem.

Q.1. Can you state the problem of walking over the 7-bridges in your own words? Is this a mathematics problem?

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The usual answer is “No”, if that happens, don’t contest it, but try to find out why they don’t consider it mathematics. If the answer is “Yes”, ask what kind of mathematics they think it is.

Ask students to share their attempts, and whether anyone succeeded. Some will think they did. Ask them to present their solutions – this will help clarify any remaining misconceptions about the problem statement.

Look at the following picture.

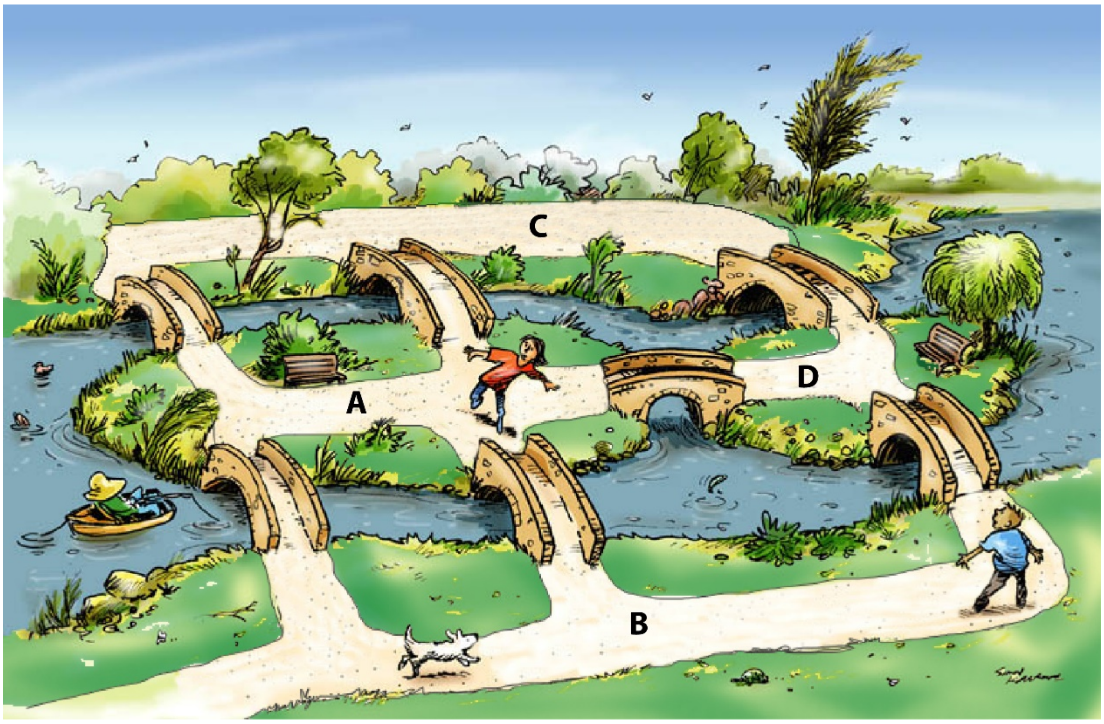

Figure 2

Look at the picture given above. *A*, *B*, *C*, and *D* represent land-masses or islands and *a*, *b*, *c*, *d*, *e*, *f* and *g* represent bridges.(Bridges are not marked in the given figure) Suppose you are asked to find a path which covers all bridges but crossing each bridge exactly once while missing no bridge.

Q.2. Can you find the required path? Share your paths with your friends.

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Q.3. Is this picture same as the one you saw of Konigsberg Bridges? Why do you think so?

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Once students respond to these questions on their papers, discuss with students whether original picture is essential to the problem? Do factors like the width of the river or the distances between bridges matter? Discuss whether a simpler drawing makes it easier to work with the problem. Ask them to further simplify as much as they can. Note that this alternate drawing is deliberately chosen to still be a physical non-abstract one. We want to see if the students can make the jump to abstract map-like representations themselves. When we worked with students, they highlighted how connection between the land and the bridges and within bridges is maintained as it is. We built on this idea of how to keep the structure of the problem same, but still simplify it.

Think about further simplifying this picture. Remove the details not required to solve the problem? Draw your simplified pictures here, and discuss with your partner how your picture/diagram still represents the problem of 7-bridges of Konigsberg.

Here individual students can present their attempts to the whole class by calling out the label of the location they want to start from and then the labels of the bridges they want to use. The teacher can use the poster or screen to show the progress.

At this moment the students will generally start assuming that finding the path is just a matter of patience, and they should be allowed to try paths until this optimism weakens and the students start asking “Well, can you show us the path?” or even “Are you sure there is a path?” or they say that “there is no such path”.

Remind them of a familiar related problem from their childhood puzzle about drawing a square and its diagonals without lifting the pencil or retracing any part.

Do you remember the popular childhood puzzle about drawing a square and its diagonals without lifting the pencil or retracing any part?

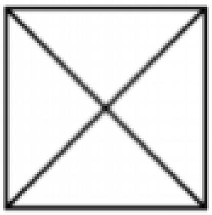

Figure 3

Q.4. Were you successful? How did you do it?

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When we did this in the class all students remembered how they were not able to solve this, and some of them drew the figure on the board using the double paper technique or folding paper technique – which basically allows them to overlap one edge, which is drawn on another paper.

The teacher can highlight the parallels between the two problems, and this becomes a starting point for believing in the possibility of a negative answer.

Task 2: Graphing the Reality!

Q. 5. See the following graph. This graph represents the same 7-bridges problem that we were working on till now. Explain how it is the same problem. Where are the bridges and lands?

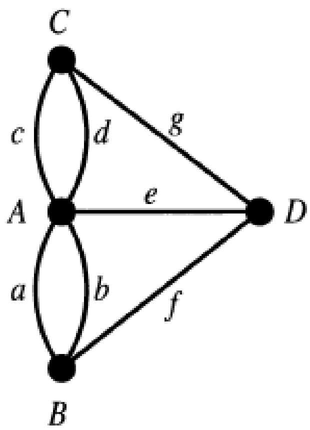

Figure 4

Give some time to students to figure out how connections between the bridges and land are preserved.

Can you trace the entire graph above without lifting your hand? Now this problem is same as the problem citizens of Konigsberg came across – waking over all the bridges once. Try here, and try with different starting points.

By now students are convinced that the tracing without lifting is not possible. Agree with the students, and talk to them how can they prove that this is not possible. At this point you might have to introduce the terminology for graphs. Start with clarifying that this graph is different from the graphs they have seen before. When we conducted this unit with students, we asked them to explain what graph they have used in the school and how this is a different graph.

A graph in Graph Theory consists of edges and vertices. The graphs are diagrams where there are vertices and lines joining any two vertices are called edges.

The number of edges, that join at a vertex are called as degree of the vertex.

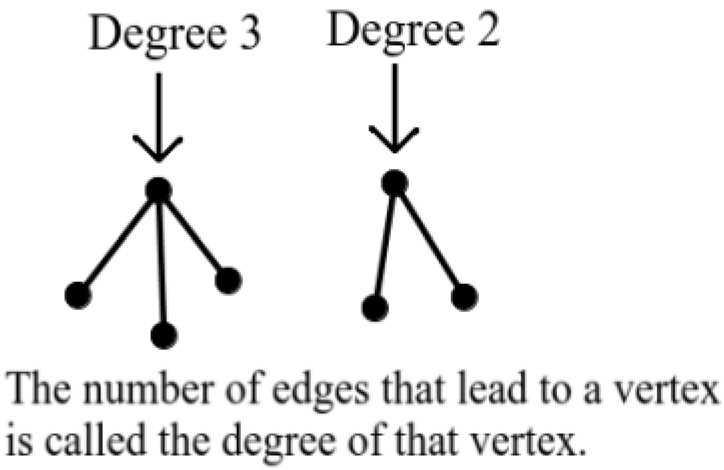

Figure 5

For example, the following diagram has 6 vertices and 7 edges, vertices A, C and E have degree 3; vertices D and F have degree 2 and vertex B has degree 1.

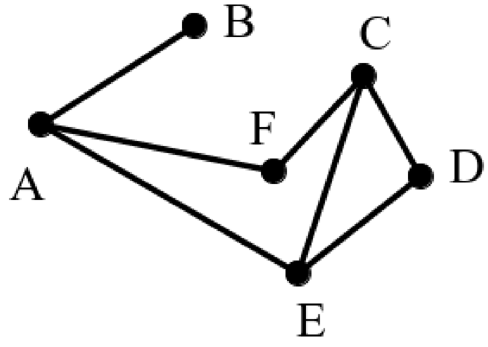

Figure 6

Draw a graph of your own; describe degree of its vertices.

Now we take up the question of how to prove a result about non-existence. Following Euler, we first try simpler examples. Here are several simple graphs. In each of them, it is possible to find a path that passes through every edge without repetition. However, not all vertices succeed as starting points for such a path. Ask students to experiment and judge which vertices are successful as starting points, when a starting point is also the final point, and whether there is any apparent connection with the number of edges at the vertices. Ask students to work in groups and record their findings first in the table.

Study the following graphs. In each of them, see whether it is possible to find a path that passes through every edge without repetition. Try different vertex as starting points.

Describe the path as sequence of letters. If you think there is no path, write no in path column. Record your findings for each graph in the following table:

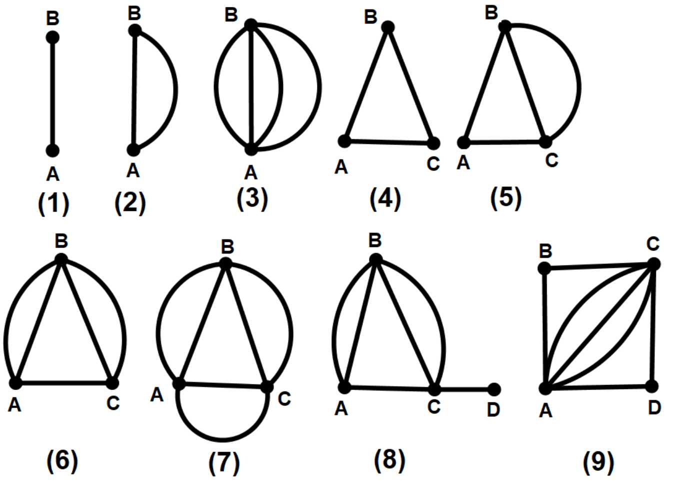

Figure 7

Record your findings for each graph in the following table:

Graph No.	Path	Are initial vertex and last vertex same? (Y/N)	Degree of initial vertex	Degree of last vertex	Degrees of other vertices
1					
2					
3					
4					
5					
6					
7					
8					
9					

Q. 6. Study the pattern carefully in the table and write your guesses about what features of the graph makes the graph traceable (crossing each edge only once) without lifting your hand. Also, if you think that some features make them non traceable, what are they?

Q.7. What pattern do you see for the graphs where the starting and ending point of the path is the same vertex? Write statements of your conjectures.

Q.8. What pattern do you see for the graphs where there is no path?

Q.9. How do you know the statements you made are true?

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Do the same exercise for the following set of graphs.

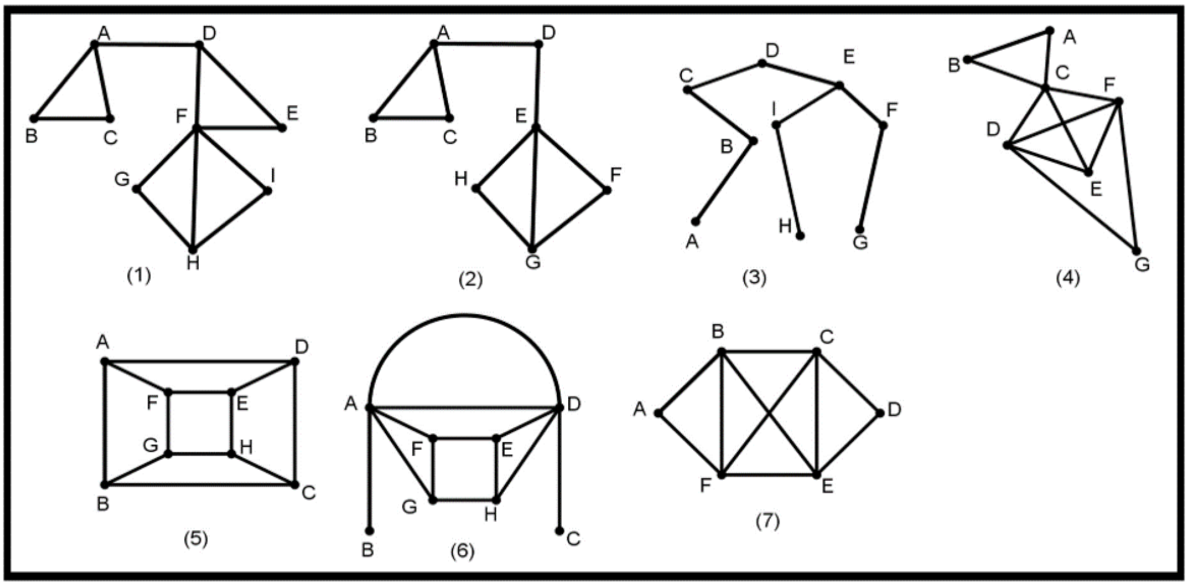

Figure 8

Were you able to find a path in all the graphs given above?

Discuss and record your findings for each graph in the following table. If you think there is no path, write no in path column.

Graph No.	Path	Are initial vertex and last vertex same? (Y/N)	Degree of initial vertex	Degree of last vertex	Degrees of other vertices
1					
2					
3					
4					
5					
6					
7					

Q.10. Study the pattern carefully in the table and write your guesses about what features of the graph makes the graph traceable (crossing each edge only once) without lifting your hand.

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Q.11. What pattern do you see for the graphs where the starting and ending point of the path is the same vertex? Write statements of your conjectures.

Q.12. How do you know the statements you made are true?

The students observe various patterns here. They observe that for even degree vertex, how many times you walk out you come back. They observe that for vertex having odd degree, there is an extra time you walk out and you do not come back. Therefore, for the graphs where you start and end at the same vertex, that vertex can have only even degree. The students also make connection between number of vertices whose degree is odd or the number of vertices whose degree is even, present in the graph.
The students at the end will learn the following.

Position of the node in the path	Degree
Node is both starting and final point of path	Even
Node is starting point but not final point	Odd
Node is final point but not starting point	Odd
Node is neither starting nor final point	Even

At this moment the teacher can help the students to define conditions for Eulerian path and cycle.

“In graph theory, a Eulerian path is a trail in a finite graph which visits every edge exactly once. Similarly, a Eulerian cycle is a Eulerian trail which starts and ends on the same vertex.”

At last, find conditions for Eulerian path and cycle in terms of number of vertices whose degree is odd. In our experiences, students themselves came up with these connections and justifications for it.

A graph is a connected graph if, for each pair of vertices, there exists at least one single path which joins them.

***Conditions for a connected graph to have a Eulerian path is that the graph must have exactly two vertices whose degree is odd.**

***Conditions for a connected graph to have a Eulerian cycle is that the graph must not have any vertex whose degree is odd.**

Coming back to Konigsberg. The citizens of Konigsberg had a hard time solving this problem. Their mayor wrote to the famous mathematician Leonhard Euler for help. And the first thing Euler did was to create a simplified and labeled drawing. Here is his drawing:

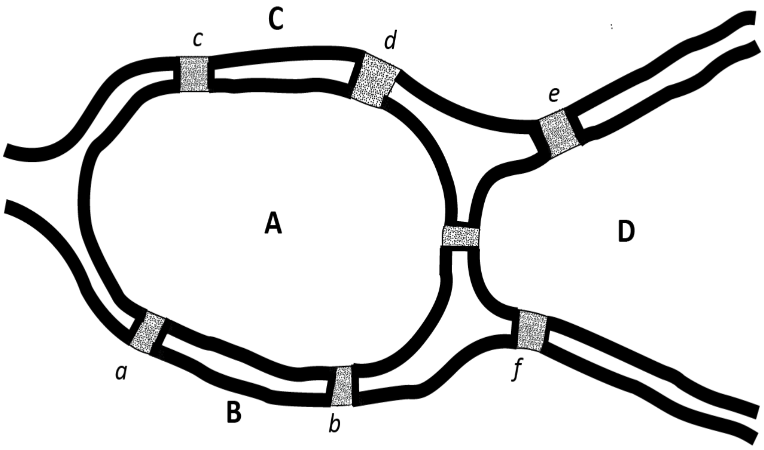

Figure 9

Q. 13. Look at this map that Euler made. How do you make sense of it? What are those letters? Compare it with the map of Konigsberg and its bridges. Can you make a graph out of this diagram?

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References:

- [BTM] Shobha Bagai, Amber Habib, Geetha Venkataraman, A Bridge to Mathematics, AGE India, 2017.
- [Mathigon] https://mathigon.org/course/graph-theory/bridges
- An online game where you can actually 'walk' across the bridges.

Image sources:

- Figure 1: https://commons.wikimedia.org/
- Figure 2: https://simonkneebone.com/tag/
- Figure 3: https://commons.wikimedia.org/

