Exploring Non Rational Numbers

Overview:

This learning unit introduces irrational numbers through the construction of lengths. In this unit, the students will be introduced to the history of irrational numbers, will construct irrational spirals, and finally construct line segments whose lengths are irrational numbers.

Time Required:

Minimum 3 periods of 40 minutes each

Type of LU:

Classroom

Materials/Facilities Required:

Pieces of string, A Geometry box, Blank sheets of paper, Multiple sets of “paper sticks” given as a printable, Pairs of scissors

Learning objectives:

(i) Understanding the concept of measuring with an appropriate unit
(ii) Understanding the existence of irrational numbers through the construction of lengths
(iii) Understanding the technique of constructing line segments of irrational lengths
(iv) Learning to partition squares to construct irrational numbers

Pre-requisites:

Pythagoras' theorem; some understanding of rational numbers; construction of right triangles

Link to Curriculum:

Class 8 -- Chapter 6, Squares and Square Roots
Class 9 -- Chapter 1, Number Systems

Suggested Readings and References:

Havil, Julian. *The Irrationals: A Story of the Numbers You Can't Count On*. Princeton University Press, 2012.
https://www.britannica.com/topic/Incommensurables-1688515
https://brilliant.org/wiki/history-of-irrational-numbers/
Maor, Eli. To Infinity and Beyond: A Cultural History of the Infinite, Birkhäuser (Boston, MA), 1987.
Shirali, Shailesh (2018) *Extending the definitions of GCD and LCM to fractions.* At Right Angles, 7 (2). pp. 33-38. ISSN 2582-1873
The Dangerous Ratio (https://nrich.maths.org/2671)

Introduction: In this learning unit, we will look at history and find out how numbers that are not rational numbers were discovered and then explore the world of these numbers by constructing them.

Part 1: Measuring using different units?

Divide the students into groups and to each group give some pieces of string. Ask the students to use this piece of string to measure the strips.

Consider two strips, Strip 1, and Strip 2 of different lengths.
Strip 1

Strip 2

Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

Strip ___ is ____ times Strip ___.

Let us look at one more pair of strips.

Strip 1 

Strip 2 

Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

After some discussions the students will understand that a piece of string is not enough. Ask them what can be used to measure both sticks. At this stage, you can give each group one set of paper "sticks" given as a printable. Ask the students to see if they can use some of these sticks to measure the strips.

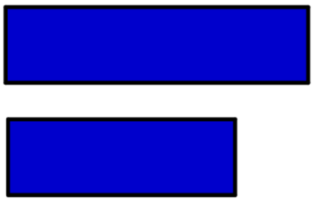
If not, let us try to measure the lengths of both strips using the "sticks" given to you. There are no markings on any of these sticks either. So, each strip needs to be a whole number multiple of the stick.

List out all the "sticks" you could use to measure both strips.

Let us do this activity with one more pair of strips.

Strip 1

Strip 2



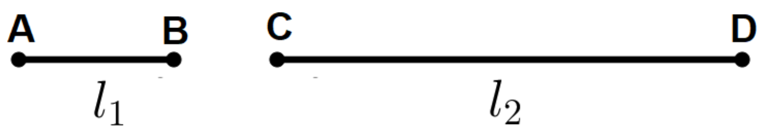
Can you use the piece of string given to you and check if Strip 1 can be used to measure Strip 2? There are no markings on the string. So, the longer strip needs to be a whole number multiple of the shorter strip.

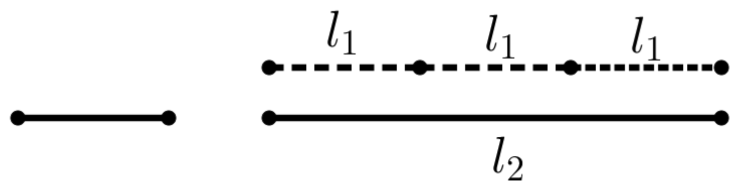
If not, let us try to measure the lengths of both strips using the "sticks" given to you. There are no markings on any of these sticks either. So, each strip needs to be a whole number multiple of the stick.

List out all the "sticks" you could use to measure both strips.

Let us try to generalise this: Instead of the strips let us look at line segments: AB and CD of lengths l_1 and l_2. If AB fits exactly a whole number of times into CD. Then we say that l_1 is a measure of l_2, and l_2 is a multiple of l_1.

Given AB and CD, two line segments with lengths l_1 and l_2 respectively,

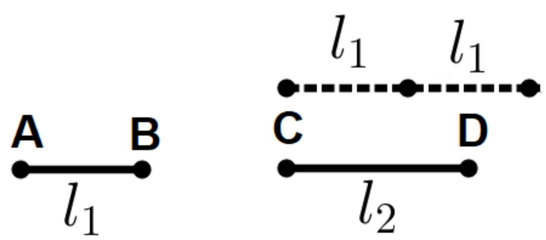




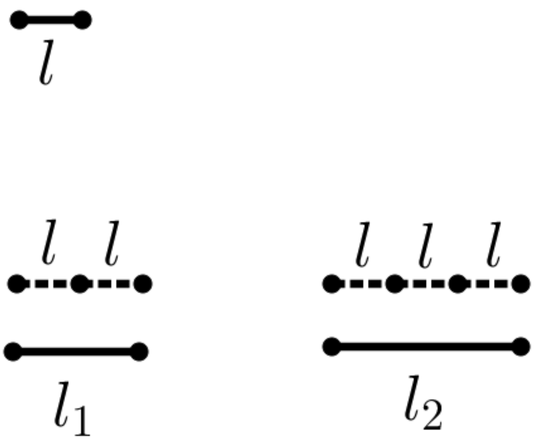
Here, we can see that, AB fits exactly 3 times into CD that means is l_2 = _____ × l_1

So, if the length of l_1 is 1 unit and the length of l_2 is _____ units.

Of course, for some other pair of line segments, it is possible that l_1 does not fit exactly a whole number of times into l_2.
For example, given two line segments AB and CD, we see that l_1 does not fit exactly a whole number of times into l_2



But, we can try to find a smaller length l, such that l fits a whole number of times in both l_1 and l_2. For example, look at the length l given below.



We see that,

l fits into l_1 exactly ______ times and l_2 exactly _____ times. (Fill in the blanks)

Such a length l is called a common measure of lengths l_1 and l_2.

And, l_1 = ____ l and l_2 = _____ l. (Fill in the blanks)

From the figure given above, $l_1 = 2\,l$, $l_2 = 3\,l$.

Can you give an example of two line segments l_1 and l_2 and one common measure l of l_1 and l_2, such that $l > l_1$ and $l > l_2$?

____

The objective of this question is to establish that any common measure is always shorter than both line segments. And one can say that there is always an upper bound for a common measure of two given line segments.

Given two line segments, l_1, and l_2 such that $l_1 > l_2$ and their common measure l, what can you say about the relationship between l_1, l_2, and l?

____

If $l_1 > l_2$, then any common measure, l is always less than l_2. And $l_1 > l_2 > l$.

Task 1

Look at the following numbers and check if one is the measure of another. If not, then try to find a common measure. Compare your answers with your friends' answers. For the remaining rows, find two pairs of l_1 and l_2, such that l_1 and l_2 are not a measure of each other.

Pair No.	l_1	l_2	Is one a measure of the other?	Common measure (l)
1	2 units	6 units	Yes	2 units, 1 unit, or 0.5 units
2	3 units	12 units		
3	5 units	3 units		
4	4 units	18 units		
5	36 units	15 units		
6			No	
7			No	

Find the greatest common measure for all the pairs of l_1 and l_2, given in the above table.
For example: For Pair 1: The greatest common measure is 2 units.
For Pair 2: The greatest common measure is _______ units.
For Pair 3: The greatest common measure is _______ units.
For Pair 4: The greatest common measure is _______ units.
For Pair 5: The greatest common measure is _______ units.
For Pair 6: The greatest common measure is _______ units.
For Pair 7: The greatest common measure is _______ units.

What can you say about the relation between the greatest common measure of two lengths and those two lengths? Discuss with your classmates.

____

The table filled by the students might look like this:

Pair No.	l_1	l_2	Is one a measure of the other?	Common measure (l)
1	2 units	6 units	Yes	2 units, 1 unit or 0.5 units, ...
2	3 units	12 units	Yes	3 units, 1.5 units, 1 unit, or 0.5 units ...
3	5 units	3 units	No	1 unit, 0.5 units, or 0.25 units ...
4	4 units	18 units	No	2 units, 1 unit or 0.5 units, ...
5	36 units	15 units	No	3 units, 1.5 units, 1 unit or 0.5 unit, ...
6			No	
7			No	

While finding the common lengths, the students might notice that there are infinitely many possible common measures. If not probe them by asking, if a certain length is a common measure, will half of it or a quarter of it also be a common measure or not? used as a unit to measure the strips.

When it comes to finding the greatest common measure, you can see that
For Pair 1, it is 2 units. For Pair 2, it is 3 units. For Pair 3, it is 1 unit. For Pair 4, it is 2 units.
For Pair 5, it is 3 units.

You can probe the students to look for a relationship between the greatest common measure and the two lengths. It can be noticed that the greatest common measure is the HCF of these two numbers.

Task 2

Fill in the following table

Pair No.	l_1	l_2	Common measures	Greatest common measure
1	1 unit	4.2 units		
2	3.5 units	2 units		
3	2.8 units	7 units		
4	$\frac{1}{3}$ unit	$\frac{1}{2}$ unit		
5	$\frac{1}{4}$ unit	$\frac{1}{6}$ unit		
6	$\frac{9}{4}$ units	$\frac{6}{5}$ units		

Compare your results with your friends.

The students filled in the table might look this:

Pair No.	l_1	l_2	Common measures	Greatest common measure
1	1 unit	4.2 units	0.2 units, 0.1units, 0.05 units, …	0.2 unit
2	3.5 units	2 units	0.5 unit, 0.1unit ,0.25 unit, …	0.5 unit
3	2.8 units	7 units	1.4 units, 0.7 unit, 0.02 unit, …	0.5 unit
4	$\frac{1}{3}$ unit	$\frac{1}{2}$ unit	$\frac{1}{6}$ unit, $\frac{1}{12}$ unit, $\frac{1}{24}$ unit, …	$\frac{1}{6}$ unit
5	$\frac{1}{4}$ unit	$\frac{1}{6}$ unit	$\frac{1}{12}$ unit, $\frac{1}{24}$ unit, $\frac{1}{48}$ unit, …	$\frac{1}{12}$ unit
6	$\frac{9}{4}$ units	$\frac{6}{5}$ units	$\frac{3}{20}$ unit, $\frac{1}{20}$ unit, $\frac{1}{40}$ unit, …	$\frac{3}{20}$ unit

In the case of numbers written using decimals, you can encourage the students to find a technique for finding the common measures for any two such numbers. One technique is to convert both numbers into natural numbers by multiplying them by the same natural number and then using the observations from the last task to get common measures (also the greatest common measure) and then dividing them by the earlier natural number.

Check the fractions that the students have selected and encourage them to use fractions such that the neither two numerators nor the two denominators are coprime. You can give them fractions like $\frac{4}{15}$ and $\frac{18}{25}$, where both the numerators and both the denominators are not coprime. These answers can help the discussion for the next task.

Now, complete the following table by taking five pairs of two fractions (where both the numerators are not equal to 1) of your choice such that *p* and *q* are co-prime numbers and *m* and *n* are co-prime numbers.

Pair No.	$\frac{p}{q}$	$\frac{m}{n}$	**Common measures**	**Greatest common measure**
1				
2				
3				
4				
5				

Task 3: Given two line segments of lengths $\frac{p}{q}$ units and $\frac{m}{n}$ units, can you find a common measure for them? Can you find a general form for their greatest common measure?

(*p* and *q* are co-prime numbers and *m* and *n* are co-prime numbers)

In these tasks, the students are first expected to write five sets of two fractions and do this exercise again for the fractions chosen by them. Ask them to observe the table and draw conclusions from them. They can see that given any two fractions, they can always find a common measure because, given two fractions, $\frac{p}{q}$ and, $\frac{m}{n}$, we can say for sure that $\frac{1}{qn}$ is a common measure. This statement can be easily proved because of the following equations.

$\frac{p}{q} = n \times p \times \frac{1}{qn}$ and, $\frac{m}{n} = m \times q \times \frac{1}{qn}$

Also, $(n \times p)$ and, $(m \times q)$ are both whole numbers when *n*, *p*, *m*, and *q* are whole numbers.

In fact, from the table, some students might be able to also find the greatest common measure of two given fractions. The greatest common measure of two fractions, $\frac{p}{q}$ and $\frac{m}{n}$ is $\frac{\text{H.C.F.}(p,m)}{\text{L.C.M.}(q,n)}$

The proof of this result can be a bit complicated so you can ask interested students to try to prove it. But do point out to the students that at this stage you have not proved the result but have made a conclusion from the table.

Proof of the greatest common measure of two fractions, $\frac{p}{q}$ and $\frac{m}{n}$ is $\frac{\text{H.C.F.}(p,m)}{\text{L.C.M.}(q,n)}$:

You can work through this with students in the following steps:

(1) You can then ask students if there is a common measure of $\frac{p}{q}$ and, $\frac{m}{n}$ which is greater than $\frac{1}{q \times n}$.

Students may notice that $\frac{1}{\text{L.C.M.}(q,n)}$ is also a common measure of $\frac{p}{q}$ and $\frac{m}{n}$ which is greater than or equal to $\frac{1}{q \times n}$.

This is because, $q \times n \geq \text{L.C.M.}(q,n)$, so $\frac{1}{q \times n} \leq \frac{1}{\text{L.C.M.}(q,n)}$

(2) You can then ask if they can find a common measure of $\frac{p}{q}$ and $\frac{m}{n}$ greater than $\frac{1}{LCM(q,n)}$.

(3) In fact, the greatest common measure for $\frac{p}{q}$ and $\frac{m}{n}$ is $\frac{\text{H.C.F.}\ (p,m)}{\text{L.C.M.}(q,n)}$.

Proof of this statement can be found here:
https://publications.azimpremjiuniversity.edu.in/1617/1/06_Shailesh_GCDandLCMtoFractions_Final.pdf

Recall, all numbers of the form $\frac{p}{q}$ or $\frac{m}{n}$, where *p*, *q*, *m*, *n* are natural numbers, are called positive rational numbers.

Given two line segments if one is a measure of another or they have a common measure then these numbers (lengths of the line segments) are called *commensurable*.

So, two line segments are called commensurable if you could find a smaller line segment that could be used as a "unit" or "ruler" (or measure or a common measure) with which you can measure both the given line segments.

In Task 3, you found a common measure of any two given fractions.

So, in Task 3, you found a very important result
"Any two positive rational numbers have a common measure, which means ***All positive rational numbers are commensurable with each other!***"

This result led to an important discovery.

Now instead of the fractions $\frac{p}{q}$ and $\frac{m}{n}$, what if we look at fractions $\frac{p}{q}$ and 1, then what can you say about their common measure?

Encourage them to see that $1 = \frac{1}{1}$ and hence the common measure of these two fractions, ($\frac{1}{1}$ and $\frac{p}{q}$) exists and is equal to $\frac{1}{q}$ in fact this is also the greatest common measure.

And hence you can say that any positive rational number $\frac{p}{q}$ is commensurable with ______.
(Fill in the blank)
Hence you can say that any positive rational number $\frac{p}{q}$ is commensurable with 1.

Part 2: Any two-line segments are commensurable?

Discovery of 'Bad' lengths

The Pythagoras theorem is named after the famous mathematician and philosopher Pythagoras who lived in the 5th Century BC (over 2500 years ago). Pythagoras founded the Brotherhood of Pythagoreans, which was devoted to the study of mathematics.

The Pythagoreans believed that given any two line segments, one is a measure of the other or you can always find a common measure for them. That is, for any two line segments either one line segment is a measure of the other or there is a third line segment which is a common measure of both the original line segments. To put it another way, they believed that whole numbers (or counting numbers), and their ratios (rational numbers or fractions), were sufficient to describe any quantity.

In general, Pythagoreans believed that "Number rules the universe.” Pythagoreans believed that underlying all spatial relations were whole numbers or at least the ratios of whole numbers (what we call positive rational numbers today).

This is the story of the discovery that the diagonal of a square of unit length could never be written as a ratio of two whole numbers. These are what we call irrational numbers today.

At the time (~500 BC), this was a very controversial idea. To illustrate this, you can also tell students about the legend of Hippasus (हिपसस), one of the Pythagoreans. The story goes that Hippasus was thrown into the sea to die because he believed that there was a line segment that was not a ratio of two whole numbers. (https://nrich.maths.org/2671)

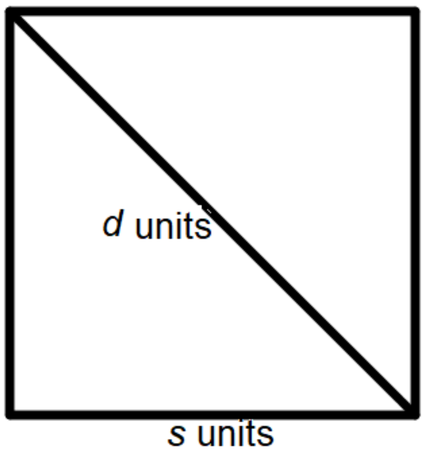
While discussing the history, make sure to let students know that these stories are interpretations of actual historical events. Not much can be said with certainty about the details of these incidents. Some of them may have no connection with actual events.

This belief was questioned when some of them found a pair of lengths that did **not** have a common measure!

Let us find out more about these two lengths.

Task 4

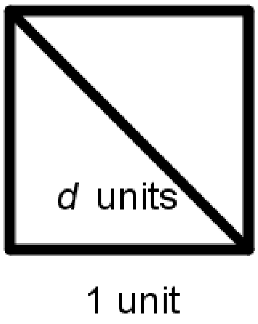
Here we have a square of side length *s* units and diagonal length *d* units.



The question was, “Do side, *s* and diagonal, *d* have a common measure?“

Now instead of any square what if we start with a square with side 1 unit?

Hint: What is the ratio of the diagonal and the side of the square? Does it change when the side changes?



Ask the students to draw different squares and find their hypotenuse using the Pythagoras theorem. They will be able to see that if the diagonal of a square with side, s is d then $d = s\sqrt{2}$

Hence the ratio of the diagonal of a square and its side is always constant.

So now the task is to find a common measure between two line segments, one of length 1 unit and another of length *d* units.

Pythagoras' Theorem tells us that for a right triangle with sides *a*, *b,* and *c* (*c* being the hypotenuse), we have
$c^2 =$ ____$^2 +$ ____^{2} (Fill in the blanks)

Applying this to the right triangle inside the square we get,

$d^2 =$ ____$^2 +$ ____$^2 =$ ____ (Fill in the blanks)

$d^2 = s^2 + s^2 = 2s^2$

Let us <u>*assume*</u> that *d* and 1 have a common measure, let us call it *l*.

Now, there exist whole numbers *A* and *B* such that $1 = A \times l$ and $d = B \times l$.

Let us assume that *l* is the greatest common measure of 1 and *d*.

$d^2 =$ ____$^2 +$ ____$^2 =$ ____ (Fill in the blanks)

$\Rightarrow B^{-} \times l^{-} = 2 \times A^{-} \times l^{-}$ (Fill in the blanks)

$\Rightarrow B^2 =$ ____ $\times A^2$ (Fill in the blanks)

$d^2 = s^2 + s^2 = 2s^2$

$\Rightarrow B^2 \times l^2 = 2 \times A^2 \times l^2$

$\Rightarrow B^2 = 2 \times A^2$

B^2 is an even number, because ____

Hence, B is an even number, because, ____

If B is not an even number, then B is an odd number but the multiplication of two odd numbers is always an odd number but B^2 is even. Hence B must be an even number.

So, let us take $B = 2B'$
where B' is another whole number

Now, $d = \_\_ \times B' \times l$ (Fill in the blank)

And, $\_\_ \times B'^2 = 2 \times A^2$ (Fill in the blank)

So, $\_\_ \times B'^2 = A^2$ (Fill in the blank)

Now, $d = 2 \times B' \times l$
And, $4 \times B'^2 = A^2$
So, $4 \times B'^2 = 2 \times A^2$

A^2 is an even number, because ____

Hence A is an even number, because, ____

If A is not an even number, then A is an odd number but the multiplication of two odd numbers is always an odd number but A^2 is even. Hence A must be an even number.

Therefore, if A is even, then we can $A = 2A'$ where A' is another whole number

Now, $1 = \_\_\_\_ \times A' \times l$ and $d = \_\_\_ \times B' \times l$
So $1 = \_\_\_ \times 2l$ and $d = \_\_\_ \times 2l$ (Fill in the blanks)

Now, $1 = 2 \times A' \times l$ and $d = 2 \times B' \times l$
So $1 = A' \times 2l$ and $d = B' \times 2l$

So, $2l$ is a common measure of 1 and d.

But, $2l$ ____ l (Fill in the blank with <, or, > or =)

$2l > l$

But we choose *l* such that it was the greatest common measure!

This contradicts what we had started with!

So, our assumption that *d* and 1 are commensurable is wrong.
You can use this opportunity to talk to the students about the technique of 'proof by contradiction:
(1) start with ONE **assumption**: 1 and *d* have the greatest common measure, *l*.
(2) arrive at an **impossible conclusion**: 2*l* is also a common measure, so *l* cannot be the greatest common measure.
(3) **reject the initial assumption**: 1 and *d* do not have a common measure

You may need to spend some time to convince the students that there is indeed a contradiction.

So, you have proved that the side of a square with side 1 unit and the diagonal of that square never have a common measure and hence are always incommensurable.

Try to modify the proof to prove that the side of any square and the diagonal of that square never have a common measure and hence are always incommensurable.

The only modification that is needed is in the initial assumption and then slight changes in the proof based on that modification for example:
Instead of the following equations: $1 = 2 \times A' \times l$ and $d = 2 \times B' \times l$
We will have, $s = 2 \times A' \times l$ and $d = 2 \times B' \times l$

But we had proved that any positive rational number is commensurable with 1 and *d* is a positive number.

So, we see that *d* is not a rational number.
Why? ___________________________________

If d was a rational number, then we have a formula to find the common measure of d and 1. But we have proved that d and 1 have no common measure hence d cannot be a rational number.

All such numbers which are not rational are called **irrational numbers**. In the tasks that follow we will construct some of these numbers and work with them.

Part 3: Constructing Irrational Numbers

Task 5

Let us try to geometrically construct line segments of different lengths which are irrational numbers. In the next tasks, we will geometrically construct line segments of different lengths which are irrational numbers.

1) Draw a right triangle such that two of its sides are of unit length. What can you say about the length of its hypotenuse?

If the students take the unit as cm, the line segment of 1 cm is too small to work with. You can ask the students to take 2 cm as their 1 unit and work.

Some students might suggest that the length of the hypotenuse can be found by measuring the length with a scale. Try to encourage them to find out the length without actually measuring it.

By Pythagoras' Theorem, we get,
Length of the hypotenuse $=\sqrt{((1)^2+(1)^2)}=\sqrt{2}$

2) Using the hypotenuse obtained in Task 5 as one leg and one leg with unit length, draw a right angle and complete the triangle. What is the length of the hypotenuse of the new triangle?

3) Continue this process for a minimum of 5 steps.

4) Draw a similar spiral starting with one of the sides of the triangle as 6 units and the other as 1 unit instead of both sides of 1 unit. Continue for a minimum of 5 steps. What can you say about the length of the last line segment you constructed?

After the students complete constructing the spiral, ask them to observe the lengths of the square roots constructed. You can bring it to their notice that as *n* increases, the difference between square roots of *n* and *n*+ 1 decreases.

5) How will you construct line segments whose length is equal to the following numbers? Give justifications as to why your answers are correct.

a) $\sqrt{32}$

There are many ways to construct a line segment of length $\sqrt{32}$ units.
i) Draw a line segment of 5 units and construct a spiral till the $\sqrt{32}$ (7 iterations)
ii) $\sqrt{32} = \sqrt{(16 \times 2)} = 4\sqrt{2}$ so $\sqrt{2}$ constructed 4 times is $\sqrt{32}$ (If you construct 4 copies of a line segment of length $\sqrt{2}$ end to end then you get a line segment of length $\sqrt{32}$)
iii)Draw a right triangle with one side of 4 units and other also as 4 units, then the hypotenuse is $\sqrt{(4^2 + 4^2)} = \sqrt{(16 + 16)} = \sqrt{32}$ units
(There are more ways than the ones given above.)
Encourage the students to justify why their methods will give the correct answer.

b) $\sqrt{40}$

i) Draw a line segment of 6 units and construct a spiral till $\sqrt{40}$ (4 iterations)
ii) $\sqrt{40} = \sqrt{(4 \times 10)} = 2\sqrt{10}$ so $\sqrt{10}$ constructed 2 times is $\sqrt{40}$ (2 copies of line segment, $\sqrt{10}$)
iii) Draw a right triangle with one side of 6 units and other as 2 units, then the hypotenuse is $\sqrt{(6^2 + 2^2)} = \sqrt{(36 + 4)} = \sqrt{40}$ units
(There are more ways than the ones given above.)
Encourage the students to justify why their methods will give the correct answer.

c) $\sqrt{50}$

i) Draw a line segment of 7 units and construct a spiral till $\sqrt{50}$ (1 iteration)
ii) $\sqrt{50} = \sqrt{(2 \times 25)} = 5\sqrt{2}$ so $\sqrt{2}$ constructed 5 times is $\sqrt{50}$
(There are more ways than the ones given above.)
Encourage the students to justify why their methods will give the correct answer.

d) $\sqrt{63}$

i) Draw a line segment of 8 units and construct a right triangle such that this line segment is the hypotenuse and the other side is 1 unit. (In some sense inverse of the spiral)
ii) Draw a line segment of 7 units and construct a spiral till $\sqrt{63}$(14 iterations)
iii) $\sqrt{63} = \sqrt{(9 \times 7)} = 3\sqrt{7}$ so $\sqrt{7}$ constructed 3 times is $\sqrt{63}$
(There must be more ways than the ones given above.) Encourage the students to justify why their methods will give the correct answer.

At this point, you can talk to the students that there are a lot of irrational numbers which we have not tried drawing like $\sqrt[3]{2}$, $\sqrt[5]{4}$ or π.

Task 6:

Draw some more triangles with irrational numbers as lengths.

1) Draw a right triangle such that its two perpendicular sides are $\sqrt{2}$ and$\sqrt{3}$ ($\sqrt{2}$ and $\sqrt{3}$ can be drawn using the techniques you found in the first part of this learning unit.) What is the length of its hypotenuse?

2) If you draw a right triangle such that the two right angle sides are $\sqrt{m}$ and $\sqrt{n}$, what is the length of the hypotenuse?

While adding irrational numbers, a lot of students think that$\sqrt{m} + \sqrt{n} = \sqrt{(m+n)}$. The main objective of this task is for the students to generalise that $\sqrt{m} + \sqrt{n} > \sqrt{(m+n)}$ for all *m* and *n* integers. If we have a right triangle with sides $\sqrt{m}$ and $\sqrt{n}$. Then the hypotenuse is always $\sqrt{(m+n)}$. Using triangle inequality, we will get $\sqrt{m} + \sqrt{n} > \sqrt{(m+n)}$.
You can also encourage the students to come up with an algebraic proof for this.

3) Is it possible to draw a right triangle whose all sides are integers? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get as side lengths? If not, give reasons.

4) Is it possible to draw a right triangle such that the two right-angle sides are integers and the hypotenuse is an irrational number? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get as side lengths? If not, give reasons.

Ask the students to show that the hypotenuse is indeed an irrational number. Ask them how they found out the length of the hypotenuse. A lot of students might at this stage find such lengths without constructing the triangles. Encourage them to find the patterns in the numbers.

5) Is it possible to draw a right triangle such that the hypotenuse is an integer and one of the other sides is also an integer and the third side is an irrational number? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

Take the hypotenuse to be 6 units. Then the square of the hypotenuse is 36. Find a partition of 36 such that one of the numbers is a perfect square and the other is not, like 16 and 20.

Now, $36 = 16 + 20$

$$6^2 = 4^2 + \left(\sqrt{20}\right)^2$$

So, the sides of the right triangle are $(4, \sqrt{20}, 6)$

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples. A lot of students might at this stage find such lengths without constructing the triangles. Encourage them to find the patterns in the numbers.

One idea is to pick any perfect square, say n^2 then partition this number in such a way that one of the numbers is a perfect square and the other one is not. This is always possible for $n > 1$.

n^2, $(n - 1)^2$ and $n^2 - (n-1)^2$ is one such example for $n > 3$. So, we get, n, n-1 and $\left(\sqrt{n^2 - (n-1)^2}\right)$are the sides of such triangles.

6) Is it possible to draw a right triangle such that the two right-angle sides are irrational numbers and the hypotenuse is an integer? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

A lot of examples of such triangles can be found without construction. For example: Take the hypotenuse to be 6 units. Then the square of the hypotenuse is 36. Find a partition of 36 such that both the numbers are not perfect squares like 14 and 22.

Now, $36 = 14 + 22$

$$6^2 = \left(\sqrt{14}\right)^2 + \left(\sqrt{22}\right)^2$$

So, the sides of the right triangle are$\sqrt{14}, \sqrt{22}, 6$.

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.

The one of ideas is to pick any perfect square, say n^2 then partition this number in such a way that both the numbers are not perfect squares. This is always possible for $n > 1$, like (n, $n^2 - 2$, n^2) one such example. Then n^2, $\left(\sqrt{n^2 - 2}\right)$ and $\sqrt{2}$ are the sides of such triangles.

7) Is it possible to draw a right triangle such that the hypotenuse is an irrational number and one of the other sides is also an irrational number and the length of the third side is an integer? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

A lot of examples of such triangles can be found without construction. For examples:

Take the hypotenuse to be $\sqrt{26}$ units. Then the square of the hypotenuse is 26. Find a partition of 26 such that one of the numbers is a perfect square and the other one is not like 16 and 10. Then the triangle the lengths of sides $\sqrt{16}$ and $\sqrt{10}$.

Now, $26 = 16 + 10$ which means $\left(\sqrt{26}\right)^2 = 4^2 + \left(\sqrt{10}\right)^2$

So, the sides of the right triangle are$\sqrt{10}, 4, \sqrt{26}$. You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples. The one idea is to pick any number which is not a perfect square and then partition this number in such a way that one of the numbers is a perfect square and the other one is not. This is always possible for *n*>1.

8) Is it possible to draw a right triangle whose all sides are irrational numbers? If yes, then draw at least two different right triangles having this property. What kind of numbers did you get? If not, give reasons.

A lot of examples of such triangles can be found without construction.
For example: Take the hypotenuse to be $\sqrt{26}$ units. Then the square of the hypotenuse is 26. Find a partition of 26 such that both the numbers are not perfect squares like 14 and 12.
Now, $26 = 14 + 12$ which means $(\sqrt{26})^2 = (\sqrt{14})^2 + (\sqrt{12})^2$
So, the sides of the right triangle are $\sqrt{12}, \sqrt{14}, \sqrt{26}$.
You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.
The one idea is to pick any number such that it is not a perfect square, say *n* then partition this number in such a way that both the numbers are not perfect squares. This is always possible for *n* > 1.

Until now we have found a lot of irrational numbers and have also drawn them. But if you notice all of them are of the type $\sqrt{n}$, where *n* is a natural number. But there are a lot of more irrational numbers like $\sqrt[3]{n}$, or $\sqrt[4]{n}$, $\sqrt[m]{n}$ or where *n* is a natural number and *m* ≥ 2 or the famous π. Do try to find out whether you can construct some of these. For which *n*'s, can you construct $\sqrt[3]{n}$, or $\sqrt[4]{n}$ or for what *m* can you construct $\sqrt[m]{n}$, for all *n*?

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