Counting Areas

Overview

In this unit, students are led through a guided discovery of Pick's Theorem and its proof. It invites them to consider the relationship between the area of special cases of grid polygons[1] and see if it can be generalised to any grid polygon or how it should be modified to be applicable to the general case. Proofs for special cases are considered and directions are given for a general case.

Learning Objectives

To provide opportunities for students to engage in mathematical practices such as

formulating conjectures,
modifying or refining them based on additional information,
generating examples to verify or refute conjectures,
proving conjectures,
considering special cases,
generalising them etc.

Material Required

Worksheet/Grid Paper

Time Required

3 sessions of 40 minutes each

Prerequisites

Students should be familiar with a basic understanding of area of polygons.

Task 1:

King Bahubali loved elephants so much that he kept a herd of them. In fact, he planted his coconut garden in such a way that it looked like an elephant when viewed from his terrace!
But the elephants would walk around the garden and destroy it. So, the king put a fence around the garden to keep the elephants away as seen in the figure given below. The trees were planted on a square grid, with one tree at each grid point, to provide sufficient space for each tree. If the king's grounds were 20 units long and 11 units wide, can you find the area available for the elephants to roam, by just counting the coconut trees?

If you cannot solve it now, go ahead with the remaining tasks, and you will be able to do this at the end of the tasks!!

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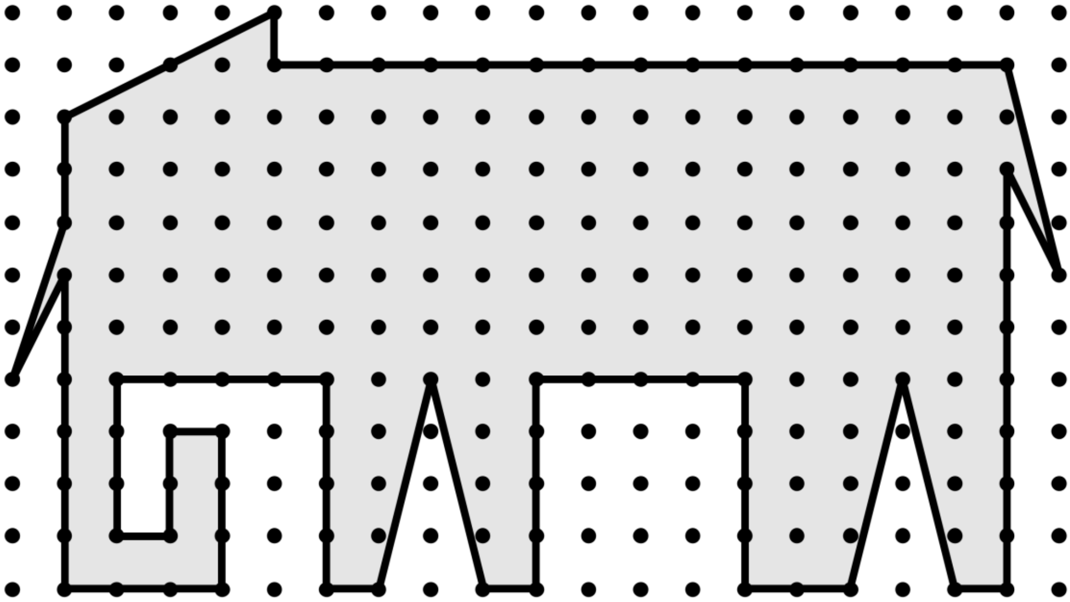
[1] Grid polygons: Polygons whose vertices are points on a square grid

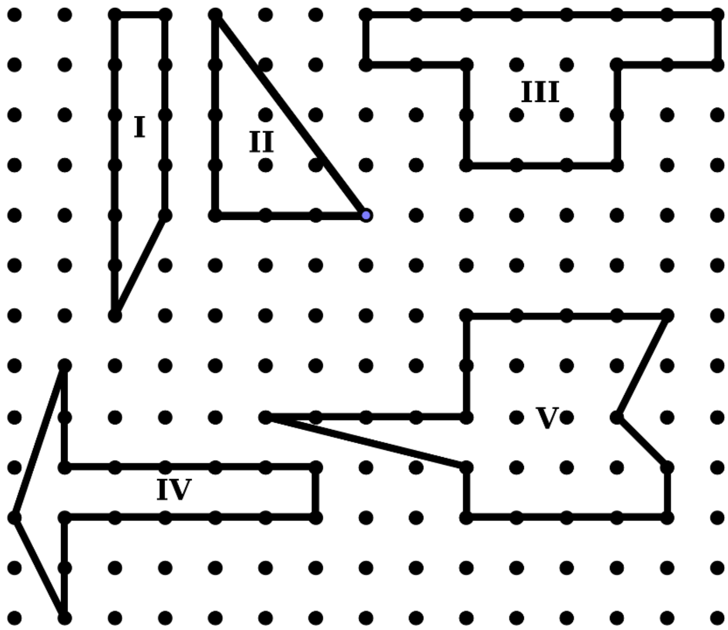


This task is meant as a motivation for the unit. Students are not expected to solve the problem at this point. They can come to the solution later, after engaging with the first few tasks of the learning unit.

Task 2:

Given below are some figures. Find the area of each and complete the given table.

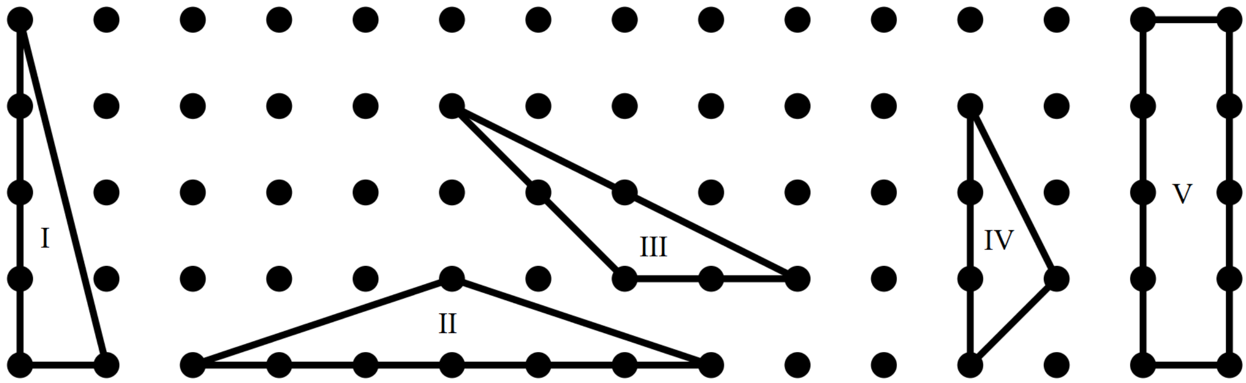
Figure	Area in Sq Units
I	
II	
III	
IV	
V	



Observe how students are finding the area of the figures. Encourage them to use multiple ways of finding areas – say counting grid squares, breaking up the figure into figures whose areas can be easily found out, or looking for 'part-squares' which add up to a complete grid-square, etc.

Task 3: Some more figures!

a) Find the area of the following figures.



Also, count the number of grid-points on the boundary of each figure, and fill the table below.

Figures	Area in Square Units	Number of grid-points on the boundary (B)
I		
II		
III		
IV		
V		

b) Do you see any relation between the area of the triangle and the number of grid-points on its boundary?

c) Does the same relation hold for figures I to V in Task 2? If not, for which ones does the relation hold?

I) The relation holds for figures ________________. (Write the number of the figure.)

ii) The relation does not hold for figures _______________. (Write the number of the figure.)

The area of the triangles = $\frac{B}{2} - 1$, where B is the number of grid-points on the boundary. The relation holds for those figures that do not have interior grid-points. The relation holds for figures I and IV of Task 2.
The table in Task 2 could be extended and appropriate columns can be added to find this out.

Task 4: Finding the expression!

a) In Task 3c), how are the figures in i) different from the figures in ii)? What property distinguishes figures in i) from figures in ii)?

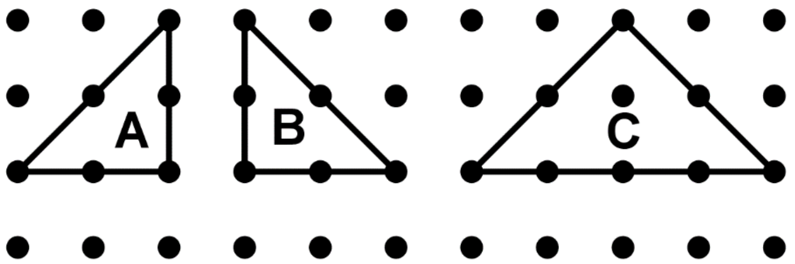
b) How would you modify the relation in Task 3b) such that it holds for all figures?

In Task 4a) the distinguishing property is that figures I and IV do not have grid-points inside them, whereas the other figures have.

Observe which are the properties that students come up with. Check if it is possible to verify that all figures that have this property satisfy the relation, $\text{Area} = \frac{B}{2} - 1$. Invite other students to provide, counterexamples of figures which do not satisfy the relation. You can provide some counterexamples if the students do not come up with any.

Provide sufficient time for students to engage with task 4 b). In case they are not able to come up with a modified relation, you may want to provide hints such as the following.

1) In the following figure, Triangles A and B, both of which have no grid-points in their interior can be put together to make triangle C, which has one grid-point in the interior.

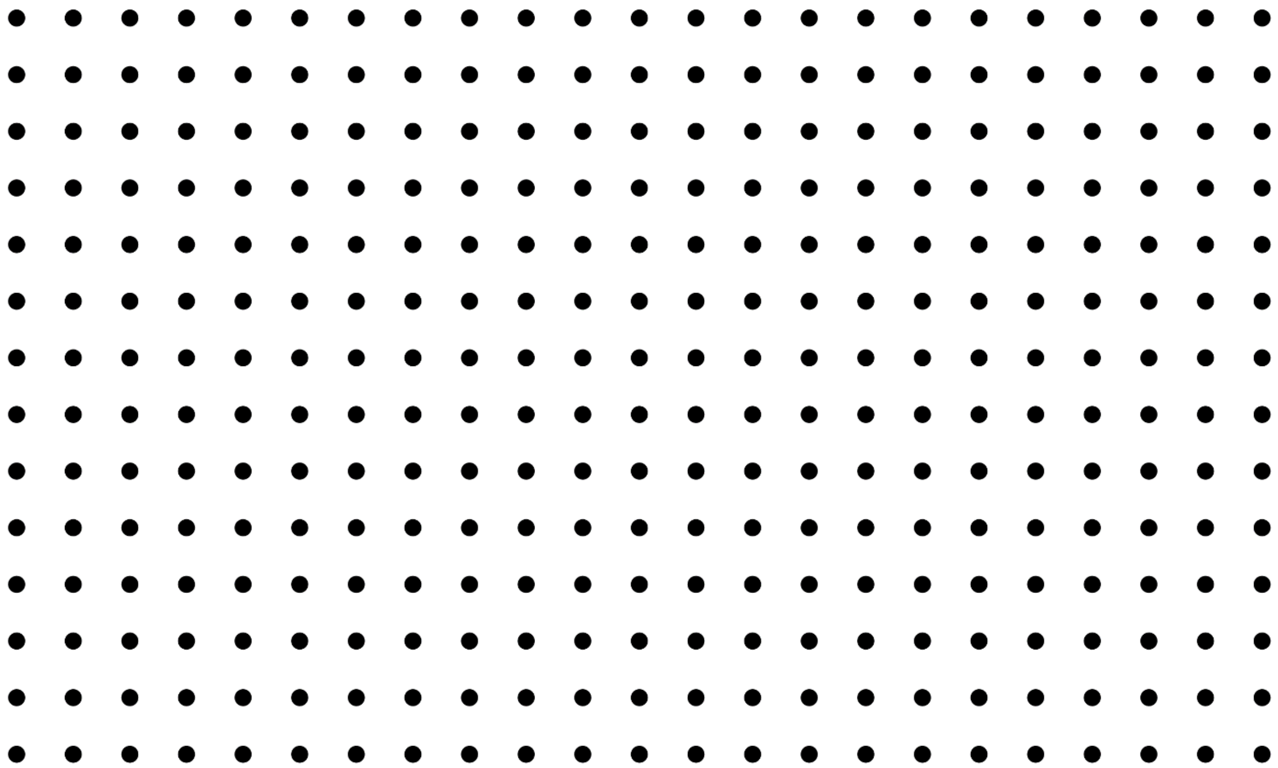


Which points on the boundary of triangles A and B and on the boundary of C? Do some points on the boundary become interior grid-points now? Do some grid-points coincide when the triangles are put together?

2) Let students tabulate the number of grid-points in the interior, on the boundary and the difference between the area and the expression $\frac{B}{2} - 1$. It can be seen that the correct formula is $\text{Area} = I + \frac{B}{2} - 1$, where I is the number of grid-points in the interior.

Task 5: Making some more figures

Draw five more figures on the grid provided below and check if the relation holds for these figures as well. Are you sure that it will hold for any figure that you may draw? What are the properties common to the figures for which this relation holds?



The relation holds for figures that have vertices on the grid points straight boundaries. That the relation holds for grid-polygons.

Did your relation hold for the figures you drew on the above grid?

We have looked at some polygons and found an expression for their area by just counting the boundary and the interior points.

Now let us look at some special polygons and prove that this expression holds for them too.

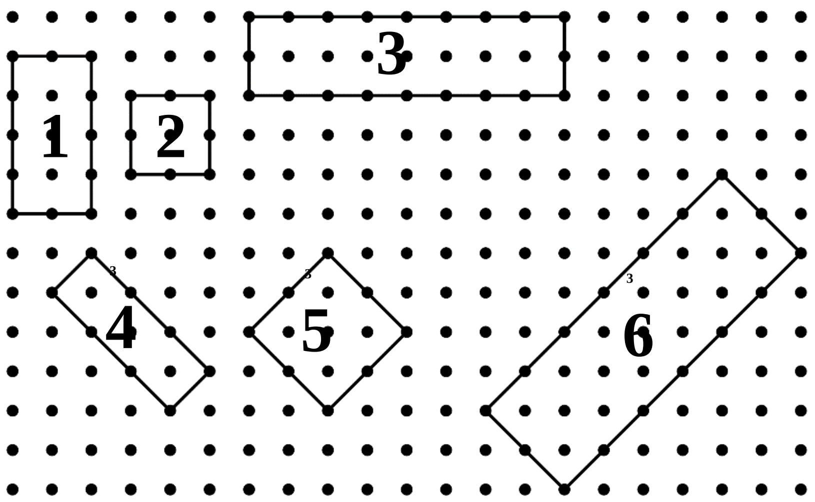
Task 6: Special cases!

Some special polygons:

In the upcoming calculations, we are going to look at some very special type of quadrilaterals, namely straight squares, and rectangles.

What do we mean by straight squares or rectangles?

Look at the rectangles given below:



In the figure: We will call rectangles 1, 2, and 3 as straight rectangles and rectangles 4, 5 and 6 as slanted rectangles. Note that Rectangle 2 is also a straight square and Rectangle 5 is a slanted square.

a) For a straight square of side *m* units

The number of grid-points in the interior (I) is ___________

The number of grid-points on the boundary (B)is ___________.

$I + \frac{B}{2} - 1$ = ___________.

How is the expression $I + \frac{B}{2} - 1$ related to the expression for the area of the square?

b) For a straight rectangle of length *m* units and breadth *n* units,

The number of grid-points in the interior (I) = ______________

The number of grid-points on the boundary (B) = ___________

$I + \frac{B}{2} - 1$ = ___________

How is the expression $I + \frac{B}{2} - 1$ related to the expression for the area of the rectangle?

For a straight square of side *m* units,

The number of grid-points in the interior (I) is $(m - 1)^2$
The number of grid-points on the boundary (B)is $4m$.

$I + \frac{B}{2} - 1 = m^2$, the area of the straight square.

This can also be used as a visual proof of the identity $(m - 1)^2 + 4m = (m + 1)^2$

In the case of the straight rectangle,
The number of grid-points in the interior (I) is $(m - 1)(n - 1)$
The number of grid-points on the boundary (B) is $2(m + n)$. $I + \frac{B}{2} - 1 = m \times n$, the area of the straight rectangle.
Some students may need to consider a few specific cases before they arrive at general expressions for the number of grid-points in the interior and boundary of a general straight square, and a straight rectangle.

For figure A, with I grid-points in its interior and B grid-points on its boundary, let us call

$I + \frac{B}{2} - 1$ as **Pick(A).**

Then for a straight rectangle and a straight square, we saw that

Pick(A) = Area(A)

This is called Pick's Theorem applied to a straight square or a rectangle.

So, we have proved Pick's theorem for a very special class of figures namely straight squares and straight rectangles. We now will go on to see if Pick's Theorem is true for all figures on the grid paper. But before that, let us go back to Task 1!

c) Complete Task 1.

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Students can now use Pick's Theorem to find the area covered by the coconut trees by counting trees and subtracting it from the area of the grounds (20 x 11 square units) to find the area where elephants can roam.

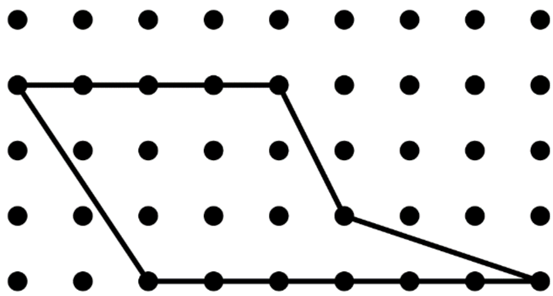
The following tasks give some hints to prove Pick's Theorem for a general grid-polygon. This can be taken up with interested students depending on the availability of time.

Task 7: What about any polygons?

We have proved that Pick's theorem holds for any straight square or any straight rectangle. But what about any polygon? In the following tasks, we will look at more such special cases and go on to prove Pick's theorem for any grid polygon.

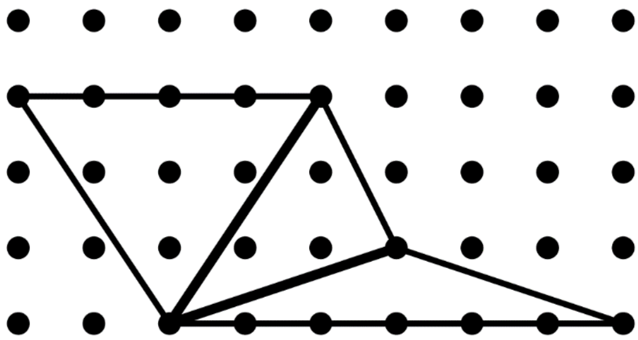
Look at the given pentagon.

a) Can you divide this pentagon into non-overlapping triangles, such that the sum of the area of all triangles is equal to the area of the pentagon? (Remember: All the vertices of each triangle should be vertices of the polygon)



How many triangles did you get?

There are many ways to do this triangulation. One of the ways is shown below.



b) Draw more polygons on your grid paper and find how many such triangles you get for each of the polygons.

This is a good time to ask them to generalize to a polygon with n vertices, that such a polygon can be divided into (n - 2) non-overlapping triangles such that all vertices of all triangles are vertices of the polygons as well. In this case, the sum of the areas of all such triangles is always equal to the area of the polygon.

We saw that any polygon can be divided into triangles. So, to prove that Pick's theorem holds for any polygon, we need to prove 2 things

1) Pick's Theorem holds for any triangle,

2) Given two shapes for which the theorem holds, it also holds for the shape formed by joining these two shapes edge-to-edge without overlap

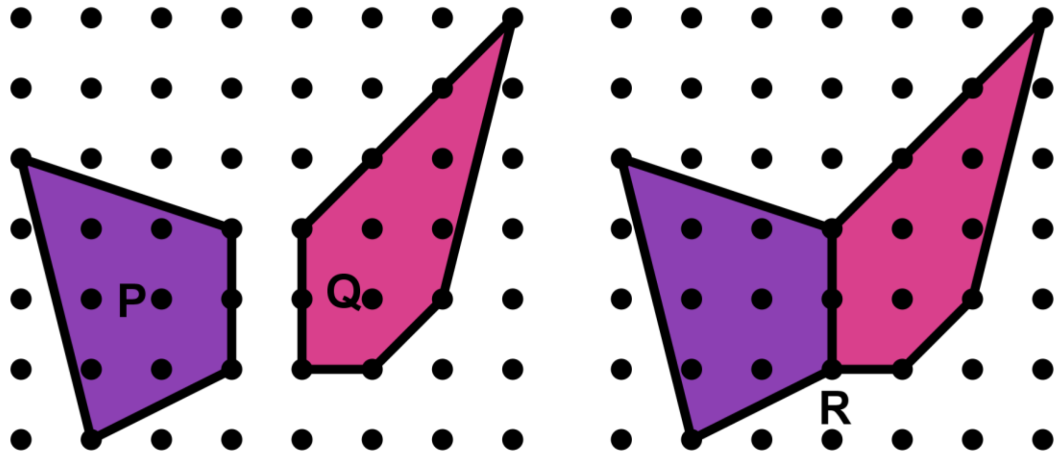
Then we can say that Pick's theorem holds for all polygons.

Spend some time on this point. Underline the point that we have broken down the problem of proving Pick's theorem for all grid polygons to two simpler problems mentioned above.

Task 8: Joining and counting!

If we put together two figures, say figure P and figure Q, in such a way that they share a boundary, and form the figure R, then we know that,

Area of (R) = Area (P) + Area (Q)



Let I_P, I_Q, and I_R be the number of grid-points in the interior of P, Q, and R respectively and B_P, B_Q, and B_R be the number of grid points in the boundary of P, Q, and R respectively.

Now, let us count I_R and B_R in terms of I_P, I_Q, B_P, and B_Q.

Let *c* be the number of grid points on the common boundary of P and Q.

Now how are the number of grid-points in the boundary of R related to those in the boundary of P and Q?

a) Can you come up with an expression for I_R and B_R in terms of I_P, I_Q, B_P, and B_Q?

$I_R = I_p + I_Q +$ ___ – ___, (Fill in the blanks) ...... (1)

$B_R = B_P + B_Q$ - ___ + ___ (Fill in the blanks) ...... (2)

(Hint: Remember the number of points of the common boundary, *c* will play an important role in this)

If *c* is the number of grid points on the common boundary,

$I_R = I_p + I_Q + c - 2$

$B_R = B_P + B_Q - 2c + 2$

Now, if we assume that Pick's Theorem holds for P and Q, then what do we get?

Area (P) = Pick (P) = _________________

Area (Q) = Pick (Q) = ________________

Now we know that Area (R) = Area (P) + Area (Q)
So,
Area (R) = Pick (P) + Pick (Q)

(Hint: Use the expressions of I_R and B_R from (1) and (2))

Area (R) = Area (P) + Area (Q) = Pick (P) + Pick (Q)

$= (I_P + \frac{B_P}{2} - 1) + (I_Q + \frac{B_Q}{2} - 1)$

$= (I_P + I_Q) + (\frac{B_P}{2} + \frac{B_Q}{2}) - 2$

$= (I_P + I_Q) + (\frac{B_P}{2} + \frac{B_Q}{2}) - 2$

Adding and subtracting (c -2)

$= (I_P + I_Q + c - 2) + (\frac{B_P}{2} + \frac{B_Q}{2} - c + 2) - 2 = (I_P + I_Q + c - 2) + (\frac{B_P}{2} + \frac{B_Q}{2} - c + 2 - 1) - 1$

$= (I_P + I_Q + c - 2) + (\frac{B_P}{2} + \frac{B_Q}{2} - c + 1) - 1 = (I_P + I_Q + c - 2) + (\frac{B_P}{2} + \frac{B_Q}{2} - \frac{2c}{2} + \frac{2}{2}) - 1$

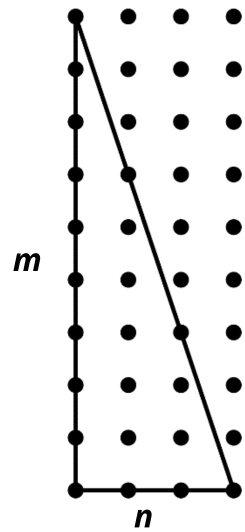
$= (I_P + I_Q + c - 2) + \frac{1}{2}(B_P + \mathrm{B_Q} - 2c + 1) - 1$

$= (I_R) + \frac{1}{2}(B_R) - 1$

= Pick (R)

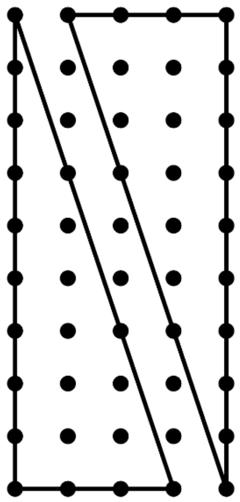
Task 9: Another special case!

In Task 7, we saw that to prove Pick's Theorem for all grid polygons we need to prove Pick's Theorem for all triangles and joining of triangles. In Task 8, we saw that Pick's theorem works for joining. So now we need to prove that Pick's Theorem holds for all triangles. But before that let us look at a very special case of triangles, namely a straight right triangle of height *m* units and base *n* units, where *m* and *n* are integers.

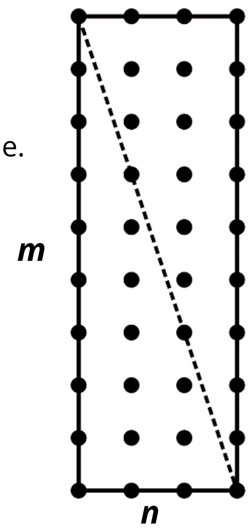


For a straight right angle of height *m* units and base *n* units, *m*, and *n* are integers,

Now we can take another congruent right triangle



And join them to make a straight rectangle.



The straight rectangle we get is of length *m* units and breadth *n* units.

From Task 6 we know that,

The number of grid-points in the interior (I) of the straight rectangle = ______________

The number of grid-points on the boundary (B) of the straight rectangle = ___________

Area (straight rectangle) $= I + \frac{B}{2} - 1$ = ___________

The number of grid-points in the interior (I) is (m - 1) (n - 1)

The number of grid-points on the boundary (B) is 2(m + n).

I + $\frac{B}{2}$ - 1 = mn, the area of the rectangle.

Now, look at (1) and (2) from Task (8) where *c* is the number of points on the common boundary

$I_R = I_p + I_Q + c - 2$

$B_R = B_P + B_Q - 2c + 2$

Also, because the triangles are congruent and symmetric on the grid, we know that here,
$I_P = I_Q$
$B_P = B_Q$
(Here P and Q are the two right triangles and R is the rectangle made by joining them.

Now let
$B_{Re} =$ Number of boundary points of the rectangle, $I_{Re} =$ Number of interior points of the rectangle,
$B_{Rt} =$ Number of boundary points of the right triangle,
and, $I_{Re} =$ Number of interior points of the right triangle,

You might have to ask the students to look at their worksheets to look at the Task 8.

So, we get,
I_{Re} = __ × I_{Rt} + c – 2 ...... (3)
B_{Re} = __ × B_{Rt} - 2c + 2 ..... (4)

I_{Re} = 2 × I_{Rt} + c – 2 ...... (3)
B_{Re} = 2 × B_{Rt} - 2c + 2 ..... (4)

We also know that Area of rectangle = ___ × Area of right triangle
And, Pick (R) = Area (R)
So, ___ × Area(Right Triangle) = Area (R) = $I_{Re} + \frac{B_{Re}}{2} - 1$
where, $B_{Re} =$ Number of boundary points of the rectangle, and
$I_{Re} =$ Number of interior points of the rectangle,
Use (3) and (4), and check
___ × Area of (Right Triangle) = Pick (R) = ____ × Pick(P)

So, Area (Right Triangle) = Pick (Right Triangle)

Pick (R) = Area (R)
2 × Area(Right Triangle) = Area (R) = $I_{Re} + \frac{B_{Re}}{2} - 1$
2 × Area(Right Triangle) = 2 × I_{Rt} + c – 2 + $\frac{1}{2}$ (2 × B_{Rt} - 2c + 2) – 1
(where, B_{Rt} = Number of boundary points of the right triangle and
I_{Rt} = Number of interior points of the right triangle
= 2I_{Rt} + c – 2 + B_{Rt} - c + 1 – 1
= 2I_{Rt} + B_{Rt} – 2
= $2(I_{Rt} + \frac{B_{Rt}}{2} - 1)$
= 2Pick (Rt)

Task 10

Until now we have checked that Pick's Theorem holds for straight squares, rectangles, and right triangles. We also looked at how Pick's Theorem holds even if you join two shapes edge-to-edge without overlap.

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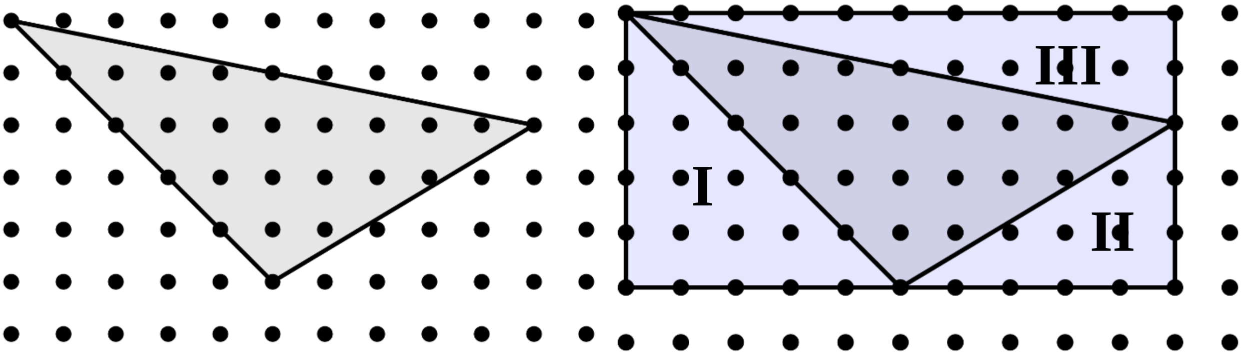
Look at the figure given below and find out what else do you need to show to prove Pick's Theorem for all polygons.



From the given diagram, one can see that any triangle can be enclosed in a straight rectangle and the remaining shapes in the rectangle will be straight right triangles.

In the above tasks, we have looked at Pick's Theorem for straight rectangles, straight right triangles and we also checked that Pick's holds when you join two figures.

Now, we need to prove that,

Pick(Rectangle) = Pick(triangle we need) + Pick (Right triangle I) + Pick(Right triangle II) + Pick (Right triangle III) and rearrange to get,

Pick(triangle we need) = Pick(Rectangle) - Pick (Right triangle I) - Pick(Right triangle II) - Pick (Right

An outline of the proof can be found here

https://nrich.maths.org/5441

or here

https://en.wikipedia.org/wiki/Pick%27s_theorem

In this unit in order to motivate the theorem, we have considered triangles which have no interior points and came up with the relation for this special case and then generalised it to other grid-polygons. In proving the theorem also, we have proved it for the special cases of a straight rectangle and a straight square and provided hints for the general proof. We started with a special case and moved to increasingly general cases.

This is just one of the ways of motivating the theorem and proof. You may want to consider other ways like

*Motivating the theorem itself through special polygons - rectangle and square
* Drawing grid-polygons with increasing number of interior points – starting from say 0, through 1, 2, 3, …. and observing the relation between the area and the number of grid-points on the boundary and interior
* Observe the relation between area and the number of grid-points on the boundary and interior of general grid-polygons
* Or any other track that the students may find engaging.

The key idea is to have students observe and find patterns, come up with examples to verify or refute this conjecture and go on to prove it.

It is also a good idea to think of extending the task – For example, some of the questions that could be explored here are
* Would the theorem still hold if there are 'holes' in the polygon? How would one need to modify the theorem (if possible) to accommodate this case?

* Would the theorem still hold if some of the boundaries of the figure are curved? How would one need to modify the theorem (if possible) to accommodate this case?

* In case there are some curved boundaries, would it be possible to 'cover' the figure with a grid-polygon and thus come up with upper and lower bounds for its area?

Notice that we are considering more and more general cases of figures on a grid and exploring if the theorem holds for these figures. This is also an aspect of mathematics that could be conveyed to students through this unit. Invite students to propose variations and ask their own questions to extend the task. Even if the solutions to all extensions proposed are not found out, the exercise of thinking through these variations may itself be valuable.

References

https://kurser.math.su.se/pluginfile.php/15491/mod_resource/content/1/picks.pdf
http://www.geometer.org/mathcircles/pick.pdf
Ian Stewart (1992) - Another Fine Math You've Got me into , Dover Publications