LU 8.14. Is your polygon the same as mine?

Overview

In this Learning Unit, students will explore the minimum conditions needed to construct a unique triangle or a quadrilateral, and other polygons. The objective of this Learning Unit is for the students to realize for themselves that the conditions which enable drawing a unique polygon are the same as the conditions of congruency that they have studied.

Minimum time required

Triangle activity (Tasks 1 to 5)– Two sessions of 40 minutes each
Quadrilateral activity (Task 6)– Two sessions of 40 minutes each
Extension to other polygons (Tasks 7 to 9)– One session of 40 minutes

Type of Learning Unit: Classroom

Introduction

Have you ever wondered how you would describe a triangle that is in your mind to somebody over the phone? What do you really say? Do you mention the sides or the angles? And would that person get the exact same figure that you had in mind? Moreover, how can you do this by giving minimum information? Today we will try to answer these questions by investigating some examples, making observations, and verifying or refuting these observations.

Unit-specific objectives

- To establish a connection between congruence and the construction of unique triangles
- To find the number of conditions necessary to ensure congruence of triangles, quadrilaterals, and other polygons
- To understand why certain sets of conditions are not criteria for congruence

Links to curriculum

- Congruence of Triangles (NCERT Mathematics Textbook Class 7) ,
- Practical Geometry (NCERT Mathematics Textbook Class 8),

Prerequisites

Students should be familiar with:

- Use of geometric tools like scale and compass
- Basic construction of triangles and quadrilaterals

Materials

Blank sheets, pencils, erasers, Geometry-boxes (Geometry compass, set-squares, protractor, and scale), scissors.

Task 1: Drawing your triangle

Q1. Draw a triangle of your choice on the given blank sheet of paper. Measure the sides and the angles of the triangle, and label the vertices of the triangle.

Before the students begin working on this activity, make sure they are sitting in pairs.
Each task requires discussing with another student, and

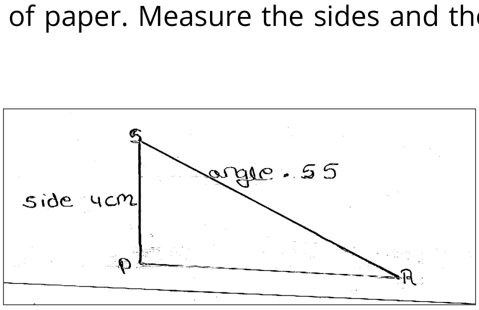

Figure T1: *Students' way of labelling a triangle*

> in this unit, we often refer to these students in pairs as partners. Make sure they all have required materials or at least that every pair has one set of scale and compass.
> This activity will give you a chance to see whether students can draw triangles, measure sides and angles, and also, whether they know how to label a triangle. See Figure T1 for an example of a student writing the angle of a triangle adjacent to a side instead of a vertex when we conducted this activity with Class 8 students.

Q2. Now see the triangles drawn by your friends. Do you see anything interesting? What is it?

Keep the paper on which you drew your triangle safely aside; we will be coming back to this triangle later in the activity.

Task 2a: Constructing a triangle when only one side is given

Draw a triangle on the given paper, one of whose side is 6 cm. Label the vertices of your triangle.

> Students might ask a question, whether all the sides of the triangle are 6 cm, and a productive answer to this question is to say "No". For the purpose of our activity, it is best not to get equilateral triangles. You can respond to them as, "One of the sides of the triangle is 6 cm, and not all". Despite this, a student might draw an equilateral triangle, and we discuss later, how to handle such a situation.

Now study the triangle drawn by your partner.

Q1. Is your triangle the same as your partner's?

Q2. How did you compare these two triangles?

Q3. You and your partner, both were told that one side of the triangle is 6 cm. Did you both get exactly the same triangles? Why?

> Make sure that you collect some oral responses here. The students can give various criteria that they used to compare the triangles; like lengths of sides or measure of the angles. When you move around, check if there are pairs of students whose triangles are obviously different. Ask these students whether for such cases they need to measure the sides or angles or they can decide using some other ways.
>
> In some cases, the two triangles being compared may look obviously different. In some other cases, one may need to check by measuring the sides and/or the angles. Another way to compare is to superimpose the two triangles, by cutting them out, or by holding the sheets of paper against a bright light. Teachers may use the opportunity to point out that when the triangles are the same (i.e., have corresponding sides and angles of the same measure or overlap exactly), then we say that the triangles are congruent.

Task 2b: Constructing a triangle when only one angle is given

Draw a triangle on the given paper where one of the angles measures 55^o. Name your triangle.

> Students might want to know the measures of the other angles. Reiterate that only one angle has been specified. Use this opportunity to revise the properties of triangles, and how, given two angles, one can always find the third angle.

Now study the triangle drawn by your partner.

Q1. Is your triangle the same as your partner's?

Q2. How did you compare these two triangles?

Q3. You and your partner, both were given one angle of 55^o. Did you both get exactly the same triangles? Why?

Task 3: Constructing a triangle when two measures are given

> Give Group A the measure of two sides, Group B the measure of one side and one angle, and Group C the measure of two angles.

Make three groups among yourselves. If possible, form your group with your classmates who are sitting close to you. Let us call these groups A, B and C.

Group A: Draw a triangle whose sides are 7 cm and 5 cm. Label the vertices of your triangle.

Group B: Draw a triangle whose one side is 6 cm and one angle is 55^0. Label the vertices of your triangle

Group C: Draw a triangle whose two angles are 50^0 and 75^0. Label the vertices of your triangle

Now study the triangle drawn by your partner.

Q1. Is your triangle the same as your partner's?

Q2. How did you compare these two triangles?

Groups A, B and C:

Q3. Did each of you get exactly the same triangles as the members in your group ? Why

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> Students most probably will verify whether the two triangles are the same or not by measuring the sides and angles of their triangles, as mentioned earlier. The teacher can also suggest superimposing (cutting the two triangles and placing on each other to see whether they overlap each other exactly), to see whether the triangles obtained are congruent to each other.

Task 4: Constructing a triangle when three measures are given

> The class is already divided into 3 groups. Now divide each group into 2 sub-groups. A1 & A2, B1 & B2 and C1 & C2. Make sure that any two students sitting beside each other are part of the

same group.

Group A1: Draw a triangle XYZ such that XY = 4 cm, YZ = 6 cm, and XZ = 7 cm.
Group A2: Draw a triangle ABC such that, AB = 5 cm, BC = 6 cm, and $\angle ACB = 45^0$.
Group B1: Draw a triangle IJK such that $\angle IJK = 40^o$, $\angle JKI = 65^o$, and $\angle KIJ = 75^o$.
Group B2: Draw a triangle STU such that $\angle UST = 50^o$, ST = 3 cm, and $\angle STU = 65^o$.
Group C1: Draw a triangle EFG such that EF = 7 cm, FG = 9 cm, and $\angle GEF = 90^{o.}$
Group C2: Draw a triangle PQR such that PQ = 5 cm, $\angle PQR = 50^o$, and QR = 4 cm.

Ask the groups to construct triangles as given in the instructions. Before they start, ask them to predict in which sub-groups would the triangles drawn be identical to one another , and in which sub-groups would the triangles be different from one another. Remember, students in group A2 should get different triangles (because this corresponds to the SSA condition, which is not a condition for congruence) and so should students in group B1 (which has the AAA condition). You might have to have a common discussion if everybody's triangles in these groups are the same to encourage the students to discover that the triangles need not be congruent even if they fulfil the conditions. The A2 group's condition (SSA) is especially interesting because there are only two possible triangles that fulfil this condition (see figure T2.) The students in groups A2 and B1 will discover that the conditions given to them were non-congruence conditions. Following this, you may point out the similarity between the conditions in A2 and C1 (both are SSA). Ask the students if the condition given in C1 is a congruence condition. Ask them to explain why.

Now study the triangle drawn by your partner.

Q1. Is your triangle the same as your partner's?

Q2. How did you compare these two triangles?

Task 5: Minimum conditions for the construction of a unique triangle

Q1. If you want your friend/partner to construct exactly the same triangle like the one you drew in Task 1, what minimum information will you have to provide, such that she/he will also construct the exact same triangle?

Q2. In the previous question, is there a different set of information that could be provided to construct the exact same triangle? Try and mention all such different sets of information that would work.

Allow students to make all kinds of possible conjectures. One common conjecture the students come up with is: give all three angles and all three sides. In response to this, you can remind them to come up with a **minimum** set of information. The six things they just mentioned is a lot of information. Again, giving three sides and two angles is the same as giving three sides and three angles (triangle's angle sum is 180°). So ensure that the information given is minimum as well as independent.

Give them sufficient time. Insist that they write down their conjectures, in any language they prefer, using diagrams, using text, or any other way that helps them to convey their thinking. After 10 to 12 minutes, collect their conjectures for further discussion.

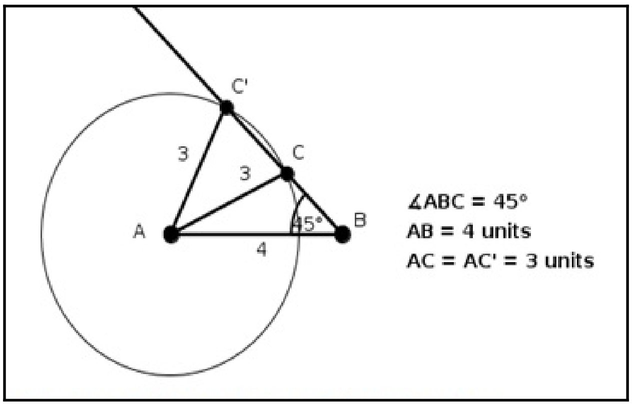

Figure T2: *Showing that SSA specification does not lead to a unique triangle*

Go through the students' conjectures, and look for possible patterns. Based on our experience, students came up with the following sets of information as leading to a unique triangle. Each may be thought of as a conjecture:

- Two sides and two angles
- All three sides (SSS)
- One side and two angles (ASA, AAS)
- Two sides one angle (SSA, SAS)
- All three angles (AAA)

Conduct a discussion about which of these will not work. Students will soon figure out that the number of minimum conditions required is three; discuss to find out which three. For some conjectures such as SSA, you might have to be ready with examples. See one such example in figure T2. Similarly, you can have counter-examples for AAA.

Students will feel more convinced about SSS, and you can use this opportunity to ask them why they are so confident about it. Conclude this activity by talking about congruency tests that work, and elicit from students the reasons why those work.

Task 6: Constructing a quadrilateral

Q1a. Now that you all know how to make a congruent triangle, let us figure out how to make a congruent quadrilateral. So if the minimum conditions for making a congruent triangle are three, what should be enough for a quadrilateral?

If the students' answer to Q1a is "four", ask them why they think it is four, and which four. Ask them if they would get a unique quadrilateral, given information, for example, about all the four sides. Then give them Q1b. At first, many students may draw squares, but when prompted to get different quadrilaterals, they will get multiple rhombuses.

Q1b. Now, given that all the sides of a quadrilateral are 3 cm, think about all the different quadrilaterals that you can draw. Draw the figures on the given blank sheet.

Q2. Did you or your partner get different quadrilaterals for Q1b?

Q3. So, if only the sides are given, is it always possible to get different quadrilaterals? How do you know?

Q4. Imagine that you have to write to your friend about a quadrilateral. Now think of the minimum information that you can send him/her, such that he/she gets the exact same quadrilateral as the one you had in your mind. What information will you send?

> Remember, in the case of quadrilaterals, you also have the diagonals. So four sides, four angles, and two diagonals, a total of ten measures form the maximum information. Give the students sufficient time to come up with combinations of information. As we ruled out four as minimum information, let us stick to five minimum measures. Collect the students' conjectures and conduct a discussion about how it would work or not work. Some examples we received while working with Class 8 learners are as follows:
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> - 4 sides and 1 diagonal
> - 2 adjacent sides and 3 angles
> - 3 sides and 2 included angles
>
> Another reasoning about why we need five pieces of information can be something like this: "To construct a quadrilateral, one needs to fix four points or vertices. By now we know that to fix three points we need three conditions (triangle). We now need to fix the fourth point. The information we have is in terms of either an angle or a length. An angle gives a straight line and a length gives us a circle. So it is clear that one condition is not enough to fix the fourth point, so we will need at least five conditions".
>
> Students can actually construct these examples and see whether they get a pair of congruent triangles. However, a general strategy to understand is to see how two congruent triangles when joined, give rise to a quadrilateral. This understanding can be used to deduce the congruency conditions for quadrilaterals. Joining two triangles reduces the information needed from six to five conditions, as one side overlaps.

Check whether what you suggested as the minimum information really works. Try drawing different quadrilaterals for the information you said you would give your friend in the question above.

Q5. Think about why this set of information will lead to congruent or non-congruent quadrilaterals.

Q6. List the conditions that worked for constructing a unique quadrilateral.

Some standard conditions that will give congruent quadrilaterals are given below for your reference.

- Three adjacent sides and two included angles within those sides
- Three angles and two included sides within those angles

  Basically, for any quadrilateral the following are the minimum conditions for congruency: AAASS, AASAS, ASASA, SASAS
- Four sides and an angle can be conditions for congruency only if the quadrilaterals are concave.

In the quadrilaterals ABCD' and ABCD'',
l(AB) = 5, l(BC) = 6, ∡ABC = 90°,
l(AD')=l(AD'') = 4 and l(CD')= l(CD'')=5.
But ABCD' is not congruent to ABCD''

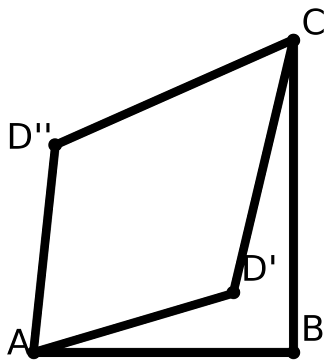

Figure T3: *Refuting a SSSSA conjecture*

Task 7: Some special triangles and quadrilaterals

We have found out the minimum information needed to draw congruent triangles and congruent quadrilaterals, but let us look at some special triangles and quadrilaterals and find out the minimum information we need to construct these.

Q1. How many conditions do you need to construct congruent equilateral triangles?

The students will find out that only the length of the equilateral triangle is enough to draw a congruent triangle. In fact, by giving the length of the side of an equilateral triangle, you are giving all the 6 pieces of information (3 sides and 3 angles). Do emphasise this while discussing this task.

Q2. How many pieces of information do you need to construct congruent squares?

The students will find out that only the length of the side is enough to draw congruent squares. In fact, by saying that it is a square and giving the length of the side of a square, you are giving all the 8 conditions (4 sides and 4 angles; all angles are 90°). Do emphasise this while discussing this task.

Q3. How many pieces of information do you need to construct congruent rectangles?

The students will find out that in the case of rectangles, you need the lengths of two adjacent sides. In fact, by giving the lengths of two adjacent sides of a rectangle, you are giving all the 8 conditions (4 sides—opposite sides are equal; 4 angles—all angles are 90º). Do emphasise this while discussing this task.

Q4. How many pieces of information do you need to construct congruent rhombuses?

The students will find out that in the case of rhombus, you need the length of one side and one angle. In fact, by giving the length of one side and the measure of one angle of a rhombus, you are giving all the 8 conditions (4 sides—all sides are equal; 4 angles—adjacent angles are complementary and opposite angles are equal). Do emphasise this while discussing this task.

Q5. How many pieces of information do you need to construct congruent parallelograms?

In the case of parallelograms, you need the lengths of two adjacent sides and the including angle. In fact, by giving the lengths of two adjacent sides of a parallelogram and the including angle, you are giving all the 8 conditions (4 sides—opposite sides are equal; 4 angles— opposite angles are equal and adjacent angles are complimentary). Do underline this while discussing this task.

Q6. How many pieces of information do you need to construct congruent trapeziums?

In the case of trapeziums, you need the length of one non-parallel side, length of the base, and measure the two bases angles. By saying that the quadrilateral is a trapezium, you can construct the other parallel side from the given information.

If you look at all the tasks together, you will notice that to construct congruent squares one needs only one piece of information explicitly, for rectangles and rhombus it will be two pieces of information, for parallelogram it is three, and for trapeziums it is four. If we recall, squares are special cases of rectangle or rhombuses, rectangles or rhombuses are special cases of parallelograms, and parallelograms are special cases of trapezium.

So as you construct more and more general quadrilaterals, you will need more pieces of information to construct unique ones, till you reach five pieces of information.

Task 8: Constructing a pentagon

Q1. Now that you all know what conditions give constructions of congruent triangles or congruent quadrilaterals, let us figure out how to construct congruent pentagons. So, if the minimum conditions for making congruent triangles are three, and that for congruent quadrilaterals are five, what do you think is the number of minimum conditions needed to construct congruent pentagons?

Mostly, the students will answer seven, by looking at the pattern. Though the answer is correct, probe the students to find some of the seven conditions, such that they give a unique pentagon.

Q2. Imagine that you have to write to your friend about a pentagon. Now think of the minimum information that you can send him/her, such that he/she gets the exact same pentagon as the one you had in your mind. What information you will send?

> As in the case of quadrilaterals, for pentagons there will be diagonals. So five sides, five angles, and five diagonals, a total of fifteen measures form the maximum information. Give the students sufficient time to come up with combinations of information. Collect the students' conjectures and conduct a discussion about how it would work or not work.

Check whether what you suggested as the minimum information really works. Try drawing different pentagons for the information you said you would give your friend in the question above.

Q3. Think about why this set of information will lead to congruent or non-congruent pentagons.

Q4. List the conditions that worked for making a unique pentagon.

> Some sets of conditions that give congruent pentagons are given below for your reference.
> - All 5 sides and 2 of the diagonals
> - All 4 sides and 3 included angles
> - 4 angles and their included sides

Task 9: Finding the number of conditions to construct a congruent polygon

Now that you know the minimum conditions needed for constructing congruent triangles, congruent quadrilaterals and congruent pentagons, let us explore how many conditions are needed for constructing congruent hexagons, or congruent heptagons.

Make some guesses, and make constructions on the given sheets of paper. Record your guesses in table 1 below.

Table 1 *Conditions required for constructing a congruent polygon*

Number of sides in the polygon	Name of the polygon	Minimum conditions required for constructing a congruent polygon
3	Triangle	3
4	Quadrilateral	5
5	Pentagon	
6	Hexagon	

7	Heptagon	
8	Octagon	

> Some students may complete the table by observing the pattern. Some students may struggle with the table. Give ample time to the students and then collect responses to complete the table on the board.

Proving our Conjectures

Let us find out how can we prove which guesses are right and which ones are wrong.
Draw a quadrilateral.
Draw a diagonal inside the quadrilateral so that it splits into two triangles.
See Figure 1.
(Here we have drawn two different types of quadrilaterals).

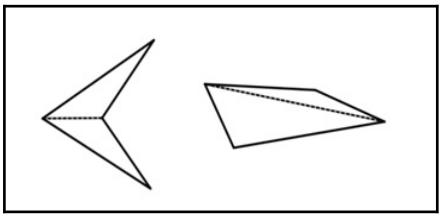

Figure 1 *Triangulation of quadrilaterals*

We see that every quadrilateral can be split into two triangles in this way. We know that for constructing a unique triangle we need three minimum conditions.
So in this case, to construct the first triangle we needed three minimum conditions. For the next triangle, we need three more, but as one side overlaps, we need only two conditions to construct a triangle congruent to the second triangle. These can be, for example, one side and the angle it makes with the adjacent side of the quadrilateral. Alternatively, one can also give two angles.

Another way of thinking about this is, once we fix the first triangle, three vertices of the quadrilateral are fixed. So to fix the remaining vertex, two conditions (as in the examples above) are sufficient. Hence these five conditions are the minimum pieces of information needed to construct a quadrilateral.

This also reconfirms our understanding of the minimum conditions needed to construct a unique quadrilateral.

What will happen if we do the same for a pentagon?

Let us draw a pentagon and see how many triangles the pentagon can be split into by drawing a minimum number of diagonals. We see from figure 2 that by drawing two diagonals, the pentagon can be split into three triangles.

For the first triangle we need three conditions, for the second triangle we need another three, but then one side overlaps so we need only two. Similarly, for the third triangle, we need two more conditions.

So, you can see that whenever you add a triangle, you add two conditions. So, the minimum conditions necessary for constructing a unique pentagon, are seven (3 + 2 + 2).

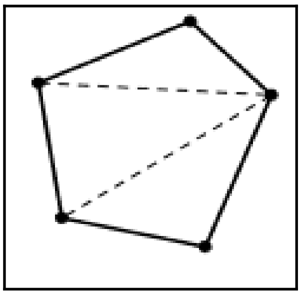

Figure 2

Let us try to figure this out for hexagons, heptagons, and octagons.

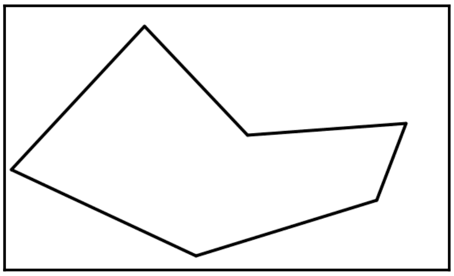

Figure 3

1. How many triangles can a hexagon be split into? (Remember that the number of diagonals drawn must be a minimum.) ________________

2. What is the minimum number of conditions needed to construct unique hexagons? ________________

3. Why?

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4. How many triangles can a heptagon be split into?

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5. What is the minimum number of conditions needed to construct unique heptagon? ________________

6. Why?

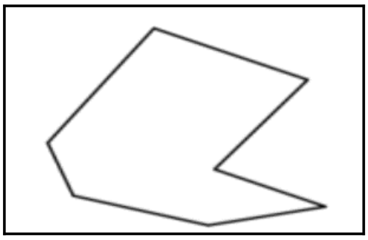

Figure 4

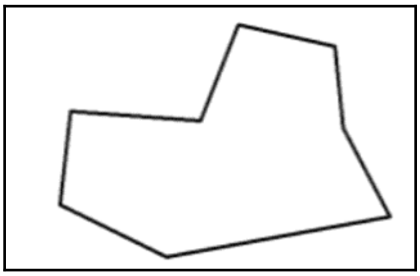

Figure 5

7. How many triangles can an octagon be split into?

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8. What is the minimum number of conditions needed to construct congruent octagon? ________________

9. Why?

____

It is interesting to see what is the minimum information needed to construct a unique polygon. Let us start with a polygon with n sides. To fix the first three vertices, we need three conditions. For each of the n - 3 remaining vertices, we need at least two pieces of information or conditions. Hence we need at least 2(n - 2) + 3 = 2n – 3 conditions. But the process of dividing the polygon into n – 2 triangles tells us that 2n -3 conditions are sufficient to construct a unique polygon. So, the number of minimum conditions required to construct a unique polygon is 2n-3.

Suggested readings

1. Simple argument about the number of minimum conditions needed to construct congruent polygons
   https://www.mathopenref.com/congruentpolygonstests.html
2. A simulation to check congruent polygons by super-positioning
   https://www.mathopenref.com/congruentpolygons.html
3. Interesting examples of congruence
   https://www.andrews.edu/~calkins/math/webtexts/geom07.htm
4. Detailed proof of why the number of minimum conditions needed to construct congruent n-gons is 2n-3
   http://www.amesa.org.za/amesal_n21_a12.pdf

References

- Plaksin, O., & Patkin, D. (2016). Exploring the minimum conditions for congruency of polygons. *Learning and Teaching Mathematics*, No.2, pp.42-46. http://www.amesa.org.za/amesal_n21_a12.pdf
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