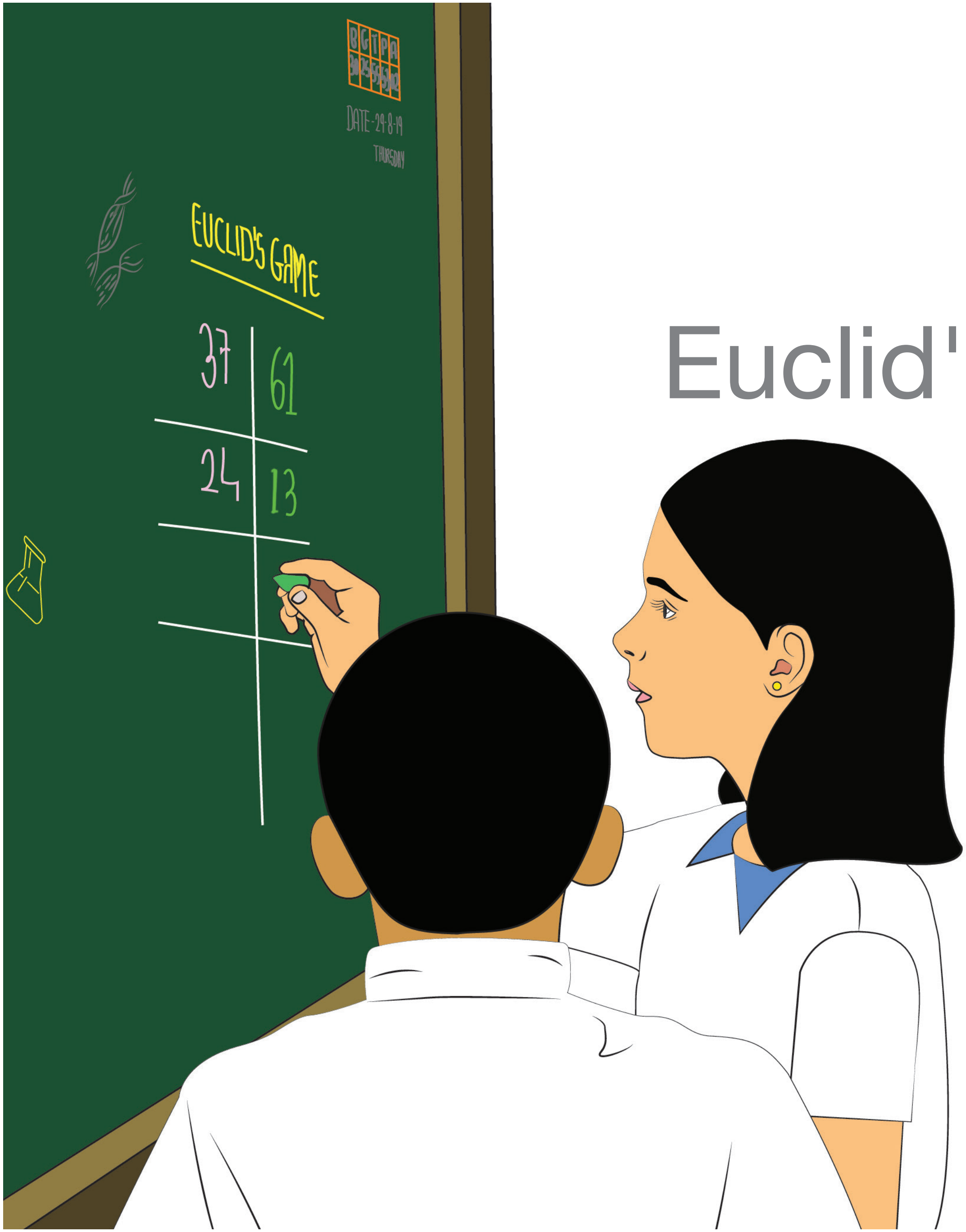


# 8.1 Euclid's game

# 8.1. Euclid's game

## Overview

Here is a game that is based on the Euclidean algorithm to find the highest common factor of two natural numbers. In this game, students not only play the game making moves that correspond to steps of the Euclidean algorithm, but also engage in doing mathematics by making conjectures, giving counter-examples, refuting conjectures, and proving some results.

An important idea is that we need to look for counter-examples to conclude that a statement is not true. As a part of this unit, you should talk to the students and try to differentiate between a true prediction and a mathematical result. Also discuss how in the case of establishing the correctness of a mathematical result, giving examples is not enough; while in the case of proving a result wrong, counter-examples are enough.

## Unit-specific objectives

- To observe patterns in numbers and articulate the observed pattern clearly
- To look for counter-examples to refute a conjecture
- To understand that examples are not sufficient to prove a conjecture
- To understand that just one counter-example is sufficient to disprove a conjecture
- To come up with a logical argument (i.e., a proof) in support of a conjecture

### Minimum time required

Three sessions of 40 minutes each

### Type of Learning Unit

Classroom

- To understand the relation between taking successive differences and the Euclidean algorithm to find the highest common factor (HCF) of two numbers

## Links to curriculum

NCERT Maths Class 6: The concepts of factors, relatively prime (co-prime) numbers and multiples (Chapter 3)

NCERT Maths Class 6: The concept of Highest Common Factor (HCF)(Chapter 3)

NCERT Maths Class 10: Euclid's Lemma (Chapter 1)

## Introduction

Playing games is a lot of fun. Today you are going to play a game that involves numbers, and you will find a way to win the game, always!

## Task 1: Play the Euclid's game

### Materials

Blackboard, chalk, sheets of paper

1. This is a two-player game.
2. The rules of the game are as follows.
   - You can decide who plays first.The first player, say Player 1, writes down a number that is between 1 and 100, including both. Let us call this number '*A*'. The second player, say Player 2, can write down another number of his/her choice. Let's call this number 'B'.
   - Now, the first player will write the number ($A - B$) or ($B - A$), whichever is positive. Let's call this number '*C*'.
   - Next, it is the second player's turn. He/she has a choice. He/She can either write the difference between *C* and *A* or the difference between *C* and *B*. However, if one of these differences is already in the list (i.e., if it is *A* or *B* or *C*) then it cannot be written again. (All differences are taken to be positive.)
   - Similarly, in subsequent turns, the players take turns to write a number which is the difference between any two numbers in the list, provided the number itself is not already present in the list.

- The game ends when it is not possible to write any new number.
- The person who writes the last number will be the winner.

Let us look at a sample run of the game.

- Suppose, the first player writes 12. The second player has 99 choices to choose his/her number (as the upper limit is 100).
- Suppose, the second player chooses 16, then the first player can only write 4, i.e. the difference between 16 and 12.
- The second player then writes 8, the difference between 12 and 4. Note that the player could not have written the difference between 16 and 4, as 12 is already in the list.
- Now there is no possibility of writing new numbers, so the game ends with the numbers 4, 8, 12, and 16 appearing in the list (12, 16, 4, 8 in the order of appearance).
- There are four numbers in the list, and the second player is the winner, as he/she wrote the last number 8.

Play this game with your partner multiple times. Study the lists of numbers that you got for each game and record your observations in the table below. For the last column, where you record the winner, mention whether Player 1 (who chose the first number) won or Player 2 (who chose the second number) won.

| Initial Numbers | | The smallest number in your sequence | The largest number in your sequence | All numbers in a sequence (in ascending order) | How many numbers are there in your sequence? | Winner |
|---|---|---|---|---|---|---|
| Player 1 | Player 2 | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |
| | | | | | | |

**Table 1**

The teacher can also make this table on the board. After all the students finish playing the game a few times, collect some responses from the students and fill in the table. (It is fun to play the game the first few times. However, the teacher should stop the game and start a discussion, when she feels that the students are getting tired or bored of repeating the game.)

Please see the section '*Mathematical and pedagogical explanations*' before recording students' responses on the board. Ensure that a sufficient number of examples that help generate some conjectures, are on the board.

The teacher can then invite the students to look for patterns in the filled table. Ask the students to generalise the patterns and make conjectures. Write down each conjecture on the board and encourage the students to examine whether it holds, by generating more examples, or can be refuted by generating a counter-example. Help the students see that verifying the conjecture for any number of examples does not amount to a proof, and that one counter-example is sufficient to disprove a conjecture. Encourage the students to come up with proofs of their conjectures.

Some of the conjectures that the students might come up with, and the ways to handle these are discussed in the section '*Mathematical and pedagogical explanations*'. However, it is possible that students come up with conjectures that are not listed here. In this case, take the conjectures one-by-one, verify, refute or prove the conjecture as the case may be, and familiarise the students with these processes.

## Task 2: Predict the numbers in the list

Let us assume that the following are the initial numbers in the game. Based on these, can you predict the numbers that you will arrive at, while playing the game?

(**Hint:** If you are stuck, look at the table you just made. See if there is any relationship between the initial numbers and the numbers in the list.)

1. Predict all the numbers in the list if:

   i. The initial numbers are 9 and 15.

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   ii. The initial numbers are 20 and 9

   _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

   iii. The initial numbers are 13 and 17.

   _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

iv. The initial numbers are 7 and 35.

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

2. How did you predict the numbers for each example? Did you notice any patterns across the examples?

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The teacher can engage students in a discussion on how they can justify that their strategy for finding the numbers in the list will always work. Please study the section '*Mathematical and pedagogical explanations*' for suggestions/ideas on how to lead this discussion.

3. Now that you know the strategy for finding the list, can you predict a strategy that will ensure that one of the players will always win this game? (Which player can adopt this strategy and always win ?)

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## Mathematical and pedagogical explanations

The activity helps the students engage in some fundamental practices of mathematics, such as observing patterns, making conjectures, verifying or refuting the conjectures, and so on. The novelty of the activity is in seeing how mathematical ideas emerge in the context of the game. The game begins with a simple task such as subtraction of two numbers. At the end of one or two games, the student begins to grasp that the process leads to a finite sequence of numbers. Moreover, one starts figuring out that the numbers that emerge in each game depend on the choice of the initial pair of numbers. After looking for patterns, the students will realise the connection between the HCF of the initial pair of numbers with the list of numbers obtained.

To understand the mathematical significance of the students' responses, we consider below the different ways of thinking that students might exhibit and propose some conjectures that they might come up with. In the pedagogical discussion column, we provide explanations for some of the conjectures, and suggestions to lead the discussion.

| Students' observations | Possible pedagogical discussion |
|---|---|
| The largest number in the sequence is the same as the largest number in the initial pair of numbers. | • Ask for the reason why the larger initial number remains the largest number in the list till the end.<br>• Explanation: As we are subtracting numbers in consecutive steps within the set of positive integers, the numbers obtained will always be smaller than the largest number in the initial pair of numbers. |
| The list contains only the multiples of the smallest number in the final list. | • Ask to verify with other examples in the table.<br>• Ask whether they can come up with a counter-example.<br>• Ask for the reason why this is the case. |
| In the initial pair of numbers, if one of the numbers is a multiple of the other, then the list contains only the multiples of the smaller initial number.<br>E.g.: For the initial numbers, 7 and 35, the sequence obtained contains only multiples of 7. | • Ask to take a few more initial pairs of numbers, where one is a multiple of the other and verify that it works every time.<br>• Look for a counter-example – a set of two numbers which will refute the above conjecture. This is an opportunity to discuss whether not finding a counter-example amounts to a proof of the conjecture. |

<table>
<tr><th>Students' observations</th><th>Possible pedagogical discussion</th></tr>
<tr><td>If the initial numbers are co-prime (i.e., 1 is the only common factor), then the list consists of all the numbers from 1 to the larger number in the initial pair of numbers</td><td>• Ask to take a few more initial pairs of co-prime numbers and verify that it works every time.<br>• Look for a counter-example – a set of two numbers which will refute the above conjecture. This is an opportunity to discuss whether not finding a counter-example amounts to a proof of the conjecture.</td></tr>
<tr><td>If the initial pair of numbers has a common factor, d, then the smallest number in the list is d.</td><td>• Note that students may not consider 1 as a common factor, and therefore while testing this conjecture they might not have co-prime numbers as the initial pair.<br>• Ask to take a few more examples of a similar kind and verify that it works every time.<br>• Ask whether the two numbers in the pair have any other common factor. Ask them whether they want to modify the conjecture. (Note that the conjecture in this particular form is not true. When we start with the initial numbers 12 and 16, 2 is a common factor of both numbers, but does not appear in the final list.)<br>• Ask them if there is anything special about the common factor that is the smallest number in the list.<br>• Think about why the smallest number in the list is this particular factor.</td></tr>
<tr><td colspan="2">If the last three conjectures above have been articulated by the students, encourage them to come up with one conjecture that will include all these three conjectures.</td></tr>
</table>

| Students' observations | Possible pedagogical discussion |
|---|---|
| The smallest number of the sequence is a common factor of the initial two numbers. | • Find out if this includes or contradicts any of the above conjectures.<br>• Refine, verify, or refute the conjecture. |
| The smallest number of the sequence is the HCF of the initial pair of numbers. | • Find out if this includes or contradicts any of the above conjectures. |

**Table T1**

Note: When students come up with observations and conjectures, these may not be articulated clearly enough. The teacher may need to rephrase, and ask clarifying questions to make the conjectures precise. There cannot be a standard instruction for this, and the teacher will have to think of ways of clarifying. Once a conjecture is formed, ask the students to verify, refute, or refine it. The next step is to think of a proof.

## Task 3: Looking for proofs of some conjectures

Some students made these interesting observations after playing a few rounds of the game:

**Observation 1:** The smallest number in the final list is the HCF of the initial pair of numbers.

**Observation 2:** All and only the multiples of this smallest number up to the largest number appear in the list.

1. Can you figure out why this happens for every pair of numbers?

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Let us look at the two observations.

- Observation 1 says the following:

a) The smallest number in the list divides both the initial numbers.

b) The smallest number is not just any common factor, but the HCF of the two initial numbers.

- Observation 2 implies the following:

a) All the numbers in the list are multiples of the smallest number in the list,

b) All the multiples of the smallest number up to the largest number appear in the list.

2. We need to prove or justify these observations. Can you think of the ways of doing this?

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The teacher could use the reasoning given below to guide the discussion with the students and to help them arrive at a proof .

**Observation 1a: The smallest number in the list divides both the initial numbers.**

**Proof for Observation 1a:** For positive numbers, whenever we subtract a number $A$ from another (larger) number $B$, the result is less than $B$. Given that we start with two initial numbers, and form subsequent numbers by subtracting the smaller from the larger, all the numbers will be smaller than the largest number in the initial pair.

Since we are not allowing negative numbers, the game has to stop at some stage. So there exists a smallest number in the list, which may be 1 or a number greater than 1.

Let us call this the smallest number $S$. If we call the initial numbers as $A$ and $B$, $A$ being the larger of the two numbers, observation 1a claims that $S$ divides $A$ and $S$ divides $B$.

Let us first prove that $S$ divides $B$.

Let assume $S$ does not divide $B$.

Then $B = nS + k$ with $k < S$ ------ By applying the division algorithm with $k$ as the remainder.

But $S$ belongs to the list. So $B - nS$, which is $k$ will also belong to the list --- we get this by subtracting $S$ "$n$ times" from $B$.

But $k < S$, which contradicts the fact that $S$ is the smallest number in the list.

That means our assumption that $S$ does not divide $B$ was wrong, so $S$ divides $B$.

Similarly, we can show that $S$ divides $A$ as well.

So $S$ is a common factor of $A$ and $B$.

**Observation 1b: The smallest number is not just any common factor, but the HCF of the two initial numbers.**

**Proof for Observation 1b:**

Let us prove a statement before we proceed with the proof of Observation 1b.

**Statement (I) : A common factor of two numbers also divides their difference.**

i.e. If $q$ divides $C$ and $D$ and $C > D$, then $q$ divides $C - D$.

**Proof of Statement (I):** $q$ divides $C$ and $D$ would mean,

Let $C = rq$ and $D = tq$ for some integers $r$ and $t > 0$

So $C - D = rq - tq = (r - t) \times q$

So, $q$ divides $C - D$ and hence is a common factor of the difference between $C$ and $D$.

**Coming back to observation 1b**

Thus, if we start with two initial numbers and $q$ is a common factor of both, it is also a common factor of their difference. This ensures that is $q$ is a common factor of all the three numbers in the list after the first step of the game. At every subsequent step, a pair of numbers is taken from the list and the difference written down as a new number. Thus if $q$ is a common factor of the existing pair of numbers, it is also a factor of the new number. This ensures that if a number is a common factor of the initial numbers $A$ and $B$, it is a factor of all the numbers in the list, including the smallest number in the list $S$. That is, any common factor of $A$ and $B$, divides $S$.

So, if the HCF of the initial numbers $A$ and $B$ is $d$, then $d$ divides $S$ and $d \leq S$. However, we know that $S$ also is a factor of $A$ and $B$ (Observation 1a). Hence $S \leq d$. Therefore, we have $S = d$.

**Observation 2a:** All the multiples of the smallest number up to the largest number appear in the list.

**Proof for Observation 2a: *S* belongs to the list, and *A* is a multiple of *S*.**

So, $A = fS$

Now $S$ and $A = fS$ are both already in the list.

So, $fS - S = (f - 1)S$ is also in the list.

Similarly, $(f - 1)S - S = (f - 2)S$ is also in the list.

Continuing like this we can see that,

**$(f - 1)S$, $(f - 2)S$, $(f - 3)S$, ..., $3S$, $2S$ also belong to the list.**

**Observation 2b:** All the numbers in the list are multiples of the smallest number in the list

Proof for Observation 2b:

Using the same argument used in the proof of Observation 1a, we get that the smallest number, $S$ divides any number in the list.

## Relation to the Euclidean algorithm

Imagine that you change the rules of the game in this way:

Instead of subtracting the smaller number from the largest, you could subtract a *multiple* of the smaller number from the larger. And then in the next step do the same with the multiple used and the number remaining after the subtraction. This then is the Euclidean algorithm for you! So, can you see why the game and therefore the Euclidean algorithm works ?

## Points to ponder

1. Do all pairs of numbers allow for a winning strategy? If not, what kinds of numbers will allow for a winning strategy?
2. What happens if you allow for the first three numbers to be random? Say, by making it a three-player game?

## Terms to discuss

Process of mathematics, conjecture, counter example, reporting a conjecture, etc.

## Suggested Readings

- Euclid's Algorithm I: https://nrich.maths.org/1357/index
- Euclid's Algorithm II: https://nrich.maths.org/1728
- Euclid's Algorithm III: https://www.cut-the-knot.org/blue/Euclid.shtml

## References

- Euclid's game: https://www.cut-the-knot.org/blue/EuclidAlg.shtml
- The optimal strategy in Euclid's game: https://math.stackexchange.com/questions/754461/optimal-strategy-in-euclids-game

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