Exploring a Dot Grid Through Rectangles

Overview
Students observe and recognise patterns. Based on their observations, they draw some conclusions. But not all the conclusions are valid. These conclusions may be valid in some cases but not in all cases. Consider the question - “What will happen to the area of the rectangle if its perimeter increases?” Many students answer that “with an increase in perimeter, the area also increases”, which is not true.

In this Learning Unit, students will explore this question and more such questions using a dot grid and construct various rectangles on it. Based on the observations, they will arrive at the conclusion that there is no obvious relationship between the area and perimeter of rectangles.

An important idea is that we need to look for counter-examples to understand statement is not true. As a part of this unit, discuss how in the case of establishing the correctness of a mathematical result, giving examples is not enough; in the case of proving a result wrong, counter-examples are enough.

These questions can be also explored using other contexts, but since the main purpose of the activity is to find counter-examples, we have chosen a specific context.

This activity may be attempted by students individually. However, discussions between students may be encouraged as they attempt the tasks.

Minimum time: Four sessions of 40 minutes each.

Type of Learning Unit: Classroom

Unit-specific objectives
- To find the areas and perimeters of rectangles on a dot grid.
- To formulate conjectures based on the areas and perimeters of rectangles.
- To understand the relationship between the area and perimeter of rectangles.

Prerequisites
Students should be familiar with basic understanding of area and perimeter of rectangles and triangles.

Introduction
You have calculated areas and perimeters of figures like rectangles and triangles, many times. Have you ever wondered what is the relationship between area and perimeter of a figure? What happens when area increases? Does the perimeter increase or decrease? What happens to the area if the perimeter decreases?

Here we will explore the relationship between the area and perimeter of rectangles. We will do this, however, with an important constraint – the rectangles will be those that can be drawn on a dot grid such that the corners (vertices) of the rectangles are grid points.

Materials
Grid papers (each student will require three to four square dotted grid papers), pencils.

Look at the grid paper you have.

Let us call the length of the line segment AB (shown in figure 1) as one unit of length.

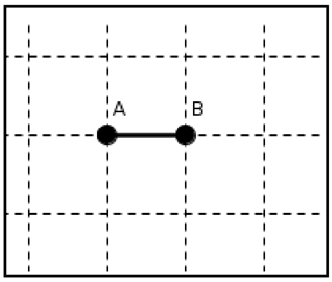

Figure 1

Ask the students what are the other lengths they can consider as units on the grid paper. Discuss what will happen if the diagonal is chosen as the unit.

If we consider the diagonal as one unit, then the length of the line segment is $\frac{1}{\sqrt{2}}$ units, which is less convenient to deal with.

Now, considering the above length as one unit, what can you say about the area of this shape in figure 2? Why do you think so? Discuss with your friends.

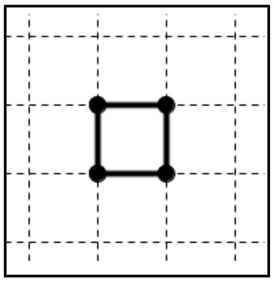

Figure 2

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At this point, do take some time out and discuss with your students about the meaning of 'the area is one square unit'. Try to emphasise the fact that any area equal to the area covered by one square of side unit length is defined to be one square unit, and is not just because the area of a square is side *x* side.

While asking the students to draw different shapes of area one square unit, ask them why they think these shapes have area one sq unit. Remember that one unit length is the distance between two adjacent dots. Some examples are given here:

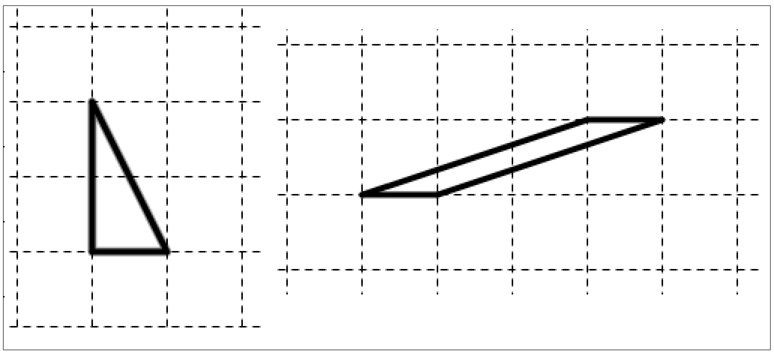


Figure T1

After the students have drawn figures of area one square unit, ask them to justify their answers.

Task 1: Any more unit areas?

Draw a few more (at least two) figures such that their areas are also one square unit.

In the following task, encourage the students to draw various rectilinear figures. The main objective of this activity is for the students to realise that area is not just a formula, and that they can actually find the area simply by counting the units squares or half-squares on the dot grid. One way to calculate the area of a rectilinear shape is shown in figure T2.

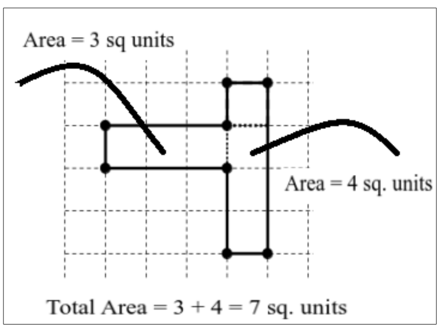


Figure T2

Task 2: When all adjacent sides are perpendicular...

The polygons drawn in figure 3 are called rectilinear polygons.

Draw five such figures and find their areas and perimeters. Remember that all the vertices of these figures should be on the grid points.

* A rectilinear polygon is a polygon, all of whose angles are either 90° or 270°. Some examples of rectilinear shapes are given here.

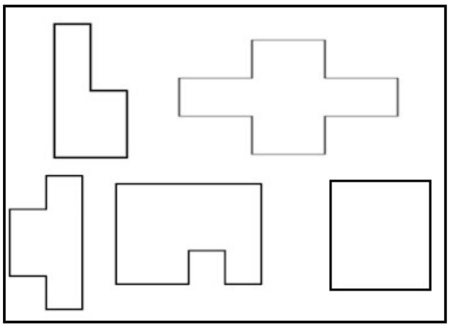


Figure 3

In the following task, try to bring to the students' attention that the figures given above can be found by cutting some triangles from or adding some triangles to a rectilinear figure. And hence one way to calculate their area is to subtract the area of the said triangles. The main objective of this activity is similar to the earlier one. This activity will encourage students further to calculate area by counting.

Also ask the students if these figures are rectilinear figures or not and let them reason why these figures are not rectilinear figures.

Task 3: What if some adjacent sides are not perpendicular...?

Figure 4 shows some non-rectilinear polygons. Find the area of the given figures.

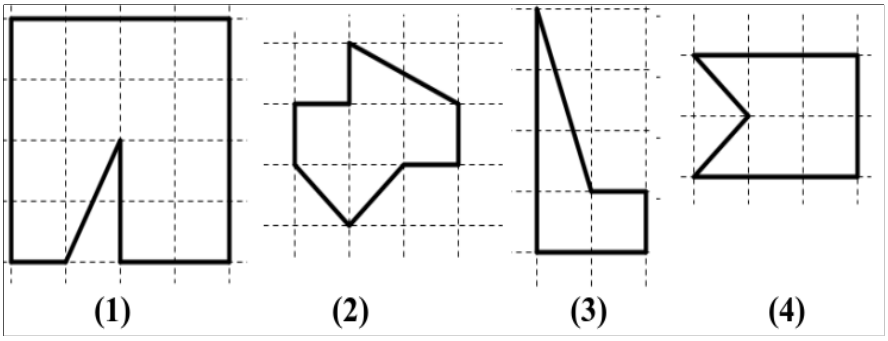


Figure 4

For the following task, encourage the students to measure areas using alternative methods. It is not necessary that everyone uses the formula to calculate area; they can count the number of unit squares inside the rectangles, or divide the rectangles into squares and add their areas, or make triangles and find the area. Different methods will also give them assurance of how the measure of the region—i.e., area remains the same even if different methods are used.

When students come up with rectangles which are not aligned to the two perpendicular axes, ask them to explore the possibility of finding the areas and perimeters of these rectangles.

One of the ways of finding the area of a tilted rectangle is to surround the rectangle with a square and remove the areas of the triangles as shown in *Figure T3*.

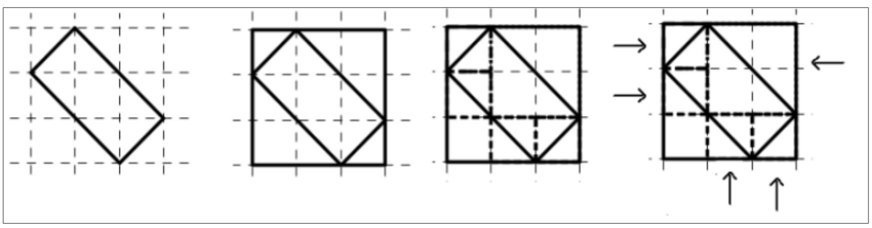


Figure T3

The students might find it difficult to find the perimeter of the rectangle in the figure T4. You can ask them to calculate the length using Pythagoras' theorem and encourage them to discuss what the length can be. This might need you to introduce to the students how square roots can be added (e.g., $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = 4\sqrt{2}$ and not $\sqrt{8}$). If some of the students suggest that $\sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} = \sqrt{8}$, ask them to justify their answer. Help them see that this is not the case. One way to do this is to square both the entities ($4\sqrt{2}$ and $\sqrt{8}$ and see that they are not equal.)

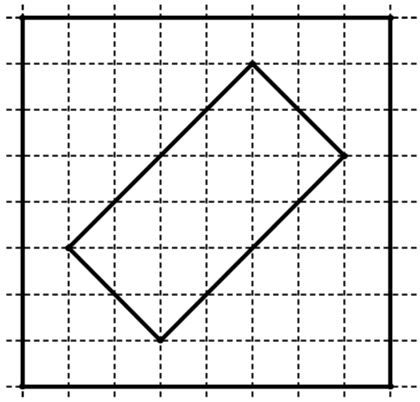


Figure T4

The important thing is to note that for any tilted rectangle on the grid, the side of the rectangle is of the type $\sqrt{n}$, where n is a natural number.

Task 4: Focusing on rectangles

Draw five rectangles on the grid paper. Keep in mind the following:
1) The vertices of the rectangles should be grid points.
2) The rectangles should be of different sizes.
3) At least one of the rectangles should be tilted.
4) Measure and write the area and the perimeter of the rectangles which are not tilted. Discuss how you got your answers.

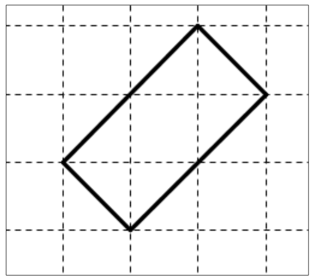


Figure 5 *Tilted rectangle*

For every sub-task in the next task, draw a table like Table 1 on the board and record the students' findings. (Table 1 provides one example for perimeter equal to 12 units.)

Table 1

Perimeter is 12 units			
Rectangle	Length	Breadth	Area
i	1	5	5
ii	2	4	8
iii	3	3	9
iv	4	2	8
v	5	1	5

While you are recording in the table, you can try to record the lengths in ascending or descending order so that the students notice a pattern in the length and the breadth. Discuss with the students which rectangles are congruent (same) and which are not. And why?

Apart from tables, you can also use graphs, or plots of lengths versus area. Plots make some of the symmetry more visible.

You can also bring to their notice that in case of perimeter 16 units, the largest area is occupied by the square (length and breadth are equal). In the case of 18, the largest area is covered when the difference between the length and the breadth is the lowest possible. Do point out that the difference of 1 is the lowest possible because we are working with whole numbers.

The two numbers 16 and 18 have been chosen, because for one of them (16) you get a square of area i.e., 16 square units, and for the other (18) you do not. You can use this as an opportunity to have a conversation about the form (4p) for a perimeter which is necessary to get a square of that perimeter. So there can be questions like, "Can there be a square of perimeter 22 on the grid?" Ask them to give reasons for their answers.

For all the sub-tasks in this unit, while you are recording in the table, you can try to record the lengths in ascending or descending order so that the children notice a pattern in the length and the breadth.

The vertices of the rectangles should be grid points!

Task 5 : Perimeter same, but rectangles different

Sub task 1

1) Draw different rectangles on your dot grid, all of which have a perimeter of 16 units. (Draw as many as you can.)
2) Complete the table given below based on your rectangles.
3) Which rectangle has the highest area? Which rectangle has the smallest area?
4) Compare your table with those of your friends.
5) Did you get a square in your table? Is a square also a rectangle? ____________________

Note: "Semi-perimeter" of the rectangle is half of the perimeter.

Perimeter is 16 units				
Rectangle	Length	Breadth	Semi-perimeter	Area
i				
ii				
...				

Sub task 2

1) Draw different rectangles on your dot grid, all of which have a perimeter of 18 units. (Draw as many as you can.)
2) Complete the table given below based on your rectangles.
3) Which rectangle has the highest area? Which rectangle has the smallest area?
4) Compare your table with the tables drawn by your friends.
5) Did you get a square in your table?

Perimeter is 18 units				
Rectangle	Length	Breadth	Semi-perimeter	Area
i				
ii				
...				

In the next task, for every sub-task, draw a table like this on the board and record the students' findings.

Perimeter is ____ units				
Rectangle	Length	Breadth	Semi-perimeter	Area
i				
ii				
...				

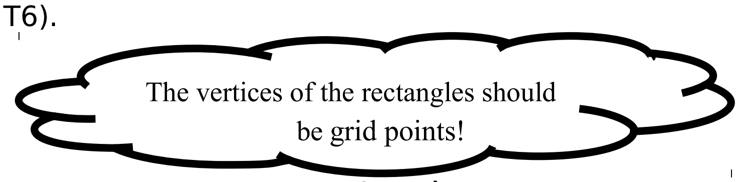
In the case of 36 square units, try to bring out the fact that the square (each side 6 units) has the lowest perimeter. And that for a given area, larger the difference between the two lengths, the greater is the perimeter.

Please note that for at least two of the sub-tasks given below, you will get at least one pair of tilted rectangles whose area would be the required area.

If the students do not come up with these examples themselves, give them clues to draw these tilted rectangles and find their areas and perimeters.

In these tasks, the students will work with 3 numbers; 36, 17, and 24. Each of these numbers gives different types of rectangles whose area is that number.

In the case of 36, you get a square (side 6 units) along with a few more rectangles; you will also get tilted rectangles whose area is 36 square units. For 17, you get 1 rectangle whose sides are parallel to the grid axis, which is definitely not a square. But you get a tilted square of area 17 square units. (see figure T5). In the case of 24, you don't get a square on the grid but get rectangles both straight and tilted. Interestingly, in case of 24, you get a tilted rectangle which is 3 copies of a tilted rectangle of area 8 square units (see figure T6).



Task 6: Now... some same area rectangles

Sub task 1

1) Draw different rectangles which have an area of 36 square units. (Draw as many as you can).
2) Fill the given table.
3) Did you get a square?
4) Compare your table with the tables drawn by some of your friends.

Area is 36 square units				
Rectangle	Length	Breadth	Semi-perimeter	Area
i				
ii				
...				

Sub task 2

1) Draw different rectangles which have an area of 17 square units. (Draw as many as you can).
2) Fill the given table.
3) Do you see any congruent rectangles among the rectangle you have drawn?
4) Compare your table with the tables filled by some of your friends.

Area is 17 square units				
Rectangle	Length	Breadth	Semi-perimeter	Area
i				
ii				
...				

Apart from the standard rectangles like 17 x 1 or 1 x 17, for area 17 square units, one can also get a tilted rectangle like the one given below.
Side of the square = $\sqrt{1^2+4^2}=\sqrt{1+16}=\sqrt{17}$

So, the area of the square is 17 square units.

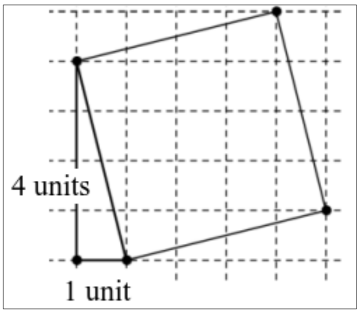


Figure T5

Sub task 3

1) Draw different rectangles having an area of 24 square units. (Draw as many as you can).
2) Fill the given table.
3) Do you see any congruent rectangles among the rectangle you have drawn?
4) Compare your table with tables filled by some of your friends.

Apart from the standard rectangles like 6 × 4, 8 × 3, 12 × 2, 24 × 1, for area 24 square units, one can also get a tilted rectangle like the one given in figure T6. Look at the square ABCD. If you count the number of unit of squares inside ABCD, you will find that there are 4 full unit squares and 8 half unit squares. So the area of square ABCD = 8 sq. units. (4 + 8 × 1/2)

Hence, the area of the rectangle ABGH is
8 + 8 + 8 = 24 square units.

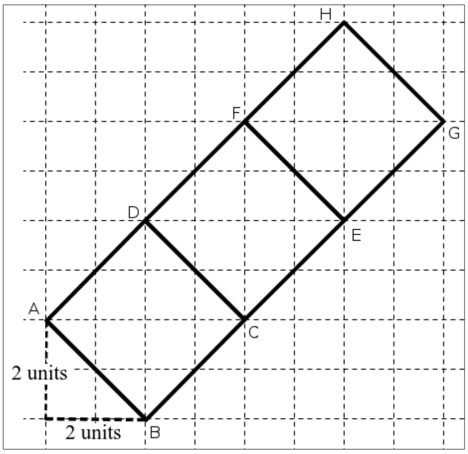


Figure T6

The vertices of the rectangles should be grid points!

Task 7 : Next, rectangles with equal areas and equal perimeters

Can you draw two different rectangles whose perimeter is 14 units and area is 12 area units? How many different rectangles did you get? Compare your rectangles with those drawn by your friends. Are they the same?

Did you get different rectangles? If yes, share your answer with your friends and your teacher. If you think it is not possible, think about why not?

After these tasks are completed, try to reach some conclusions for the following statements (More such statements can come from your students.)
1. It is possible that even if the area of a rectangle increases, the perimeter decreases.
2. It is possible that even if the perimeter of a rectangle increases, the area decreases.
3. There are rectangles which are not congruent but have the same area or the same perimeter.

4. There is no obvious relationship between the area and perimeter of a rectangle.
5. If for two rectangles, the area and the perimeter are both equal, then the rectangles are congruent.

Claim: If for two rectangles, the area and the perimeter are both equal, then the rectangles are congruent.

Proof:

Given that: Rectangle 1 and 2 have the same area and the same perimeter.

Let l_1 and b_1 be the length and the breadth of the first rectangle, respectively, and l_2 and b_2 be the length and the breadth of the second rectangle, respectively.

Given that the area of the two rectangles are equal.

So, $l_1 \times b_1 = l_2 \times b_2$ .... (I)

and $2(l_1 + b_1) = 2(l_2 + b_2)$

$l_1 + b_1 = l_2 + b_2$ .... (II)

From (I) and (II) we have,

$l_1 = l_2 + b_2 - b_1$ ------------(III)

So substituting (III) in (I),

$(l_2 + b_2 - b_1) \times b_1 = l_2 \times b_2$

So, $l_2 b_1 + b_1 b_2 - b_1^2 = l_2 b_2$

$b_1 b_2 - b_1^2 = l_2 b_2 - l_2 b_1$

Therefore, $b_1(b_2 - b_1) = l_2(b_2 - b_1)$

Hence, $(b_1 - l_2) \times (b_2 - b_1) = 0$

So, either $b_2 = \mathrm{b}_1$ or $l_2 = \mathrm{b}_1$

The two rectangles are congruent as all the sides are equal and all the angles are equal.

Let the students explore the following questions, on a grid paper. You can also ask them to check for more such patterns.

Task 8: More with the grid: possibilities and impossibilities

Remember that all the vertices of all the figures you draw should be grid points.

Use the grid paper, explore, and find answers to the following:

1) If the length and the breadth of a rectangle are natural numbers, and its area is an odd number, what can you say about that rectangle's semi-perimeter (half of the perimeter)?

If the area is odd, it means that the length and the breadth are both odd. And hence their sum, which is the semi-perimeter, is even.

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2) If the length and the breadth of a rectangle are natural numbers, and its semi-perimeter is an odd number, what can you say about the area of this rectangle?

If the semi-perimeter is odd, it means that either the length is an odd number or the breadth is an odd number, and the other is even. And hence their product, which is the area, is even.

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3) What are the different possible areas of triangles drawn on the grid? Are all the multiples of half achieved?

All the multiples of half can be achieved. For example, look at the triangles given in figure T7. If you continue this process then you can get any multiple of half.

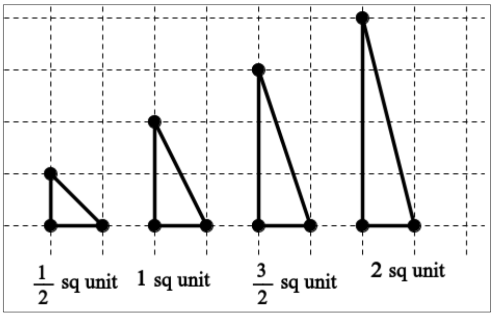


Figure T7

4) What are the possible areas of grid squares (squares with vertices on the grid)? (These may not have integer sides!)

It will be obvious to the students, that for a perfect square number one can always draw a square whose area is equal to that perfect square (side is equal to the square root). Ask the students if there are any other possibilities. You might have to give them hints to start looking at other possibilities. For example,

I: Consider a right-angled triangle of sides *a* units and *b* units (see figure T8), then the length of the hypotenuse is $\sqrt{(a^2+b^2)}$ units.

If you draw a square of side *a*+*b* units in such a way then, the side of the inscribed square is $\sqrt{a^2+b^2}$ units.

So, the area of the inscribed square is a^2+b^2 sq. units.

More importantly, the side of any grid square can be written as a square root of the sum of two squares, for example $\sqrt{a^2+b^2}$

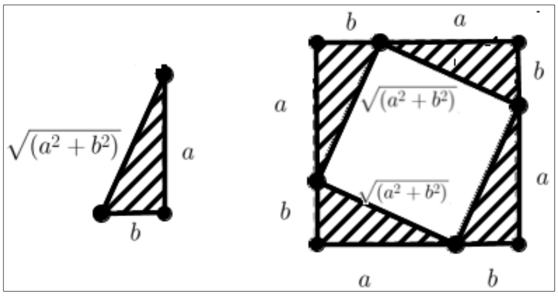
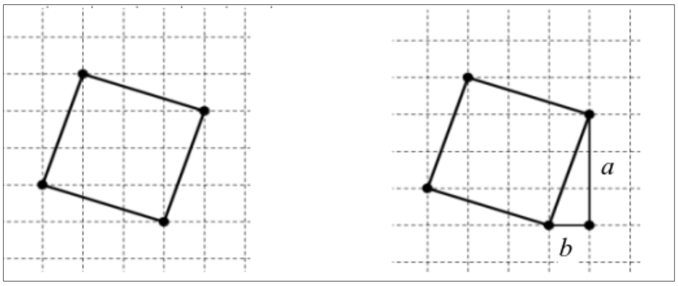


Figure T8

II. Also given any tilted square like the one in figure T9 (a), it can be seen that the side of that square is of the form, units (see figure T9 (b)).

Here it is clear that the length of the side of the given square is equal to



(a) **Figure T9** (b)

From **(I)** and **(II)**, it is clear that, for any number which can be written as the sum of two squares, one can draw a square whose area would be that number.

5) What are the possible areas of tilted rectangles (rectangles with vertices on the grid)? (Hint: Look at tilted squares first).

Suggested readings

1. An interesting exploration of area and perimeter of triangles:

De, P, Sircar, S, Titus, S (2017): LFHC - Area, Perimeter and Congruency (APC), At Right Angles-November 2017, Azim Premji Foundation (https://azimpremjiuniversity.edu.in/SitePages/resources-ara-november-2017-LFHC-area-perimeter-congruency.aspx)

2. A book which talks about Pedagogical Content Knowledge:

Ma, L. (1999). Knowing and teaching elementary mathematics: teachers' understanding of fundamental mathematics in China and the United States. Chapter 4.

3. A presentation on mathematical proofs and examples of proofs:

N.J.: Lawrence Erlbaum Associates- https://study.com/academy/lesson/mathematical-proof-definition-examples-quiz.html

4. A short course on mathematical languages and language of proofs- https://www.birmingham.ac.uk/Documents/college-eps/college/stem/Student-Summer-Education-Internships/Proof-and-Reasoning.pdf

References

- De, P., Sircar, S., & Titus, S. (2017, November). LFHC - Area, Perimeter and Congruency (APC). *At Right Angles, Volume 6,* No.3, pp.53-58. Azim Premji Foundation. https://azimpremjiuniversity.edu.in/SitePages/resources-ara-november-2017-LFHC-area-perimeter-congruency.aspx
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Lawrence Erlbaum Associates.