Exploring Irrational Numbers

Summary:
This learning unit on irrational numbers aims at introducing irrational numbers to students through constructing lengths. In this unit, the students will be introduced to the history of irrational numbers, will construct irrational spirals and finally construct line segments whose lengths are irrational numbers.

Time Required:
Minimum 3 periods of 40 minutes each

Type of LU:
Classroom

Materials/Facilities Required:
Geometry box, Blank sheets of papers

Learning objectives:
(i) Understanding the history of irrational numbers
(ii) Getting introduced to the concept of irrational numbers
(iii) Understanding the technique of constructing line segments of irrational lengths
(iv) Learning to partition squares to construct irrational numbers

Pre-requisites:
Rudimentary understanding of irrational numbers, Pythagoras theorem, Constructions of right angle triangles

Link to Curriculum:
Class 8 -- Chapter 6
Class 9 -- Chapter 1

Suggested Readings:
The Dangerous Ratio (https://nrich.maths.org/2671)
Eli Maor - To Infinity and Beyond: A Cultural History of the Infinite , Birkhauser
https://www.britannica.com/topic/Incommensurables-1688515
https://brilliant.org/wiki/history-of-irrational-numbers/
The Irrationals: A Story of the Numbers You Can't Count On. Princeton University Press.

Introduce the topic through a small thought-experiment. By thought-experiment we mean an experiment where you imagine the experiment and predict the result. But you can also use straws and ribbons to demonstrate commensurability.

Introduction

Part 1:

In this learning unit we will find out something about the history of irrational numbers and explore the world of irrational numbers through right angle triangles. Through various construction exercises, we will get familiarized with operations on irrational numbers.

Consider two ribbons, Ribbon 1 and Ribbon 2 of different lengths measured in some common units. Now I want to measure the lengths of both the ribbons using the same stick. The condition is that the the length of both the ribbons should be whole number multiples of the length of the stick, as there are no markings on the stick.

Solve some questions based on the above situation.

1) Length of Ribbon 1 = 3 cm
   Length of Ribbon 2 = 9 cm

What will be the length of the stick you will use to measure these two ribbons?
Length of the stick = ________ cm

Discuss your answers with your friends.

> Some possible answers can be 1 cm, 1.5 cm or 3 cm. At this stage you can also discuss with the students if a certain length can be used then half of it or a quarter of it can also be used as a unit to measure the ribbons.

2) Length of Ribbon 1 = 4 cm
   Length of Ribbon 2 = 6 cm

What will be the length of the stick you will use to measure these two ribbons?

Length of the stick = ________ cm

Discuss your answers with your friends.

> Some possible answers can be 1 cm, 2 cm or 0.5 cm.

3) Length of Ribbon 1 = 3 cm
   Length of Ribbon 2 = 4 cm

What can be the length of the stick you will use to measure these two ribbons?

Length of the stick = ________ cm

Discuss your answers with your friends.

Some possible answers can be 1 cm or 0.5 cm.

Now instead of the ribbons let us look at line segments. Let AB and CD be two line segments of lengths *r* and *s*. And when we compare the shorter length, *r* to the longer length, *s*, if we find that *r* fits exactly a whole number of times into *s*. Then we say that *r* is a measure of *s*, and *s* is a multiple of *r*.

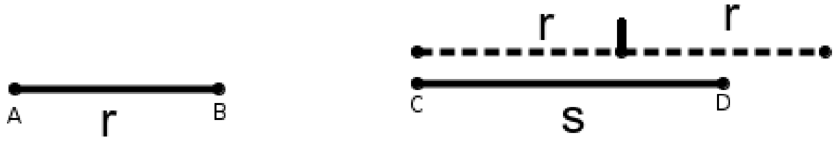
For example, length of AB is *r* and length of CD is *s*,

Then, we find that *r* fits exactly 3 times into *s* that is $s = \ldots\ldots \times r$ and s/r = __/__ (Fill in the blanks)

So we choose *r* as the unit of measurement then length of *r* is 1 unit and the length of *s* is __ units.

Of course, it is possible that *r* does not fit exactly a whole number of times into *s* but it may be possible to find a smaller length *y* that fits a whole number of times into *r* and *s* both. Then we say that *y* is a common measure of *r* and *s*.
Notice that, then the ratio of s/r may be expressed as a ratio of whole numbers.

For example, length of AB is *r* and length of CD is *s*.



Then *r* does not fit exactly a whole number of times into *s*. But can we find *y* such that *y* fits exactly whole number of times into both *r* and *s*?
Let us see how. Consider a line segment EF whose length is *y*,

So *y* fits into *r* exactly ___ times and *s* exactly ___ times.

Is *y* a common measure of *r* and *s*?

___________ (Yes/No)

And, *r* = __ *y* and *s* = __ *y*.

> You can use ribbons and straws and show the difference between r being a measure of s and r and s having a common measure by using ribbons and straws of different lengths.

Task 1:

Look at the following numbers and check if one is the measure of another. If not, then try to find a common measure. Compare your answers with your friends' answers.

Pair No.	*r*	*s*	Is *r* a measure of *s*?	Common measure (*y*)
1	2 cm	6 cm	Yes	2 cm, 1 cm or 0.5 cm
2	3 cm	12 cm		
3	3 cm	5 cm		
4	4 cm	18 cm		
5	15 cm	36 cm		

In all the pairs was *r* a measure of *s*? ______________________

Task 2:

Fill in the following table

Pair No.	Length 1 (*L1*)	Length 2 (*L2*)	Is *L1* a measure of *L2*?	Is there a common measure?	Common measure
1	1 cm	4.2 cm			
2	2 cm	3.5 cm			
3	2.5 cm	6 cm			
4	1/3 cm	1/2 cm			
5	1/6 cm	1/4 cm			
6	5/6 cm	¾ cm			

Compare your results with your friends.

Given two line segments if one is a measure of another or there is a common measure for both of them then these numbers (lengths of the line segments) are called ***commensurable***.

So in short, two line segments are called commensurable if you could find a smaller line segment that could be used as a "unit" or "ruler" (measure or a common measure) with which you can measure both the given line segments.

Task 3:

Take any two fractions (non-unit fractions) of your choice and try to find a common measure for them.

> See the fractions the students have selected and try to find fractions such that the two numerators or the two denominators are not co-prime. You can give them fractions like 4/15 and 18/25, where both the numerators and both the denominators are not co-prime. These answers can help the discussion for the next task.

Task 4: Given two line segments of lengths $\frac{p}{q}$ units and $\frac{n}{m}$ units, can you find a common measure for them?

> $\frac{1}{q \times m}$ is a common measure of $\frac{p}{q}$ and $\frac{n}{m}$:
>
> We can say that,
> $\frac{p}{q} = (p \times m) \times \frac{1}{q \times m}$ and $\frac{n}{m} = (n \times q) \times \frac{1}{q \times m}$
> Hence, $\frac{1}{q \times m}$ is a common measure of $\frac{p}{q}$ and $\frac{n}{m}$.
> And , we also know that highest common factor (HCF) of *p* and *n* is always a factor of *p* and *n* and least common multiple (LCM) of *q* and *m* is a multiple of *q* and *m*.
> Some bit of manipulations give us that $\frac{hcf\ (p,n)}{lcm\ (q,m)}$ is a common measure of $\frac{p}{q}$ and $\frac{n}{m}$.
> This result is very important because in this problem, the children have actually found the result which says that, “All rational numbers are commensurable with each other.” Pythagoras theorem is named after a famous mathematician and philosopher Pythagoras. Pythagoras gained his famous status by founding a group, the Brotherhood of Pythagoreans, which was devoted to the study of mathematics.’’
>
> While discussing the history make sure that you tell the children a lot of these stories are interpretations of information found hence very little can be said with certainty. “Pythagoreans believed that "*Number rules the universe*" and the Pythagoreans gave numerical values to many objects and ideas. Pythagoreans also believed that underlying all spatial relations were whole numbers or at least the ratios of whole numbers. The discovery which will follow made it obvious that this was not true and that there was a line segment whose length could never be written as a ratio of two whole numbers.
> At this point, you can also tell them the legend of Hippasus (हिपसस) (https://nrich.maths.org/2671) and how he was drowned in the sea because he said that there was a line segment which was not a ratio of two whole numbers.

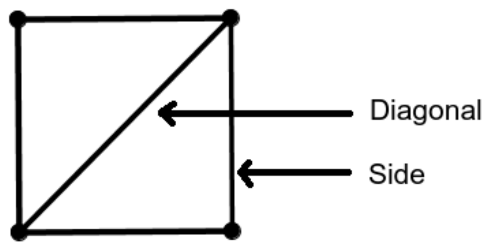
You have proved a very important result. This result shows that any rational number is commensurable with 1 unit.

You must have heard about the Pythagoras Theorem. The Pythagoras theorem is named after a famous mathematician and philosopher Pythagoras. Pythagoras gained his famous status by founding a group, the Brotherhood of Pythagoreans, which was devoted to the study of mathematics.

'*The Pythagoreans believed that given any two line segments, one is a measure of the other or you can always find a common measure for them. That is, for any two line segments either one line segment is a measure of the other or there was a third line segment which is a common measure of both the original line segments*.'

This belief was shattered when they found a pair of lengths that did NOT have a common measure. The two lengths were the side of a square and its diagonal.

Let us find out more about these lengths.



Let the length of the side be 1 unit. Then what can you say about the length of the diagonal, *s*?

Step 1:

$s^2 = 1^2+1^2 = 2$ ....Pythagoras Theorem

Let us assume that 1 and *s* have a common measure then there exists a number y such that $s = n \times y$ and $1 = m \times y$ where *n* and *m* are whole numbers. Then we get that,

$\frac{s}{1} = \frac{n}{m}$ ------- (1)

Let us assume that *y* is largest possible common measure of 1 and s.
(Notice that this is a fair assumption, as the common measure cannot be greater than the smaller of the two lengths.)
Actually the common measure has to be strictly less than 2 because $s^2 = 2$ so
s has to be between 1 and 2.
So, 1 = *my* and *s* = *ny*
Pythagoras theorem says that,

$s^2 = 1^2+1^2 = 2$

But $\frac{s^2}{1} = \frac{n^2}{m^2}$ ------------ Squaring (1)

So,

$s^2 = \frac{n^2}{m^2}$

Then, $n^2 = s^2 \times m^2 = 2m^2$

So we can say that *n* is an even number. Why?

> The explanation for this is that if *n* was an odd number then n^2 would also be an odd number but n^2 is even hence *n* has to be even.

So *n* = 2*k*, *k* is a whole number.

n^2 = ____ k^2 ..... (Fill in the blank)

But, $n^2 = 2m^2$

So, $2m^2$ = _______ ..... (Fill in the blank)

or m^2 = _______..... (Fill in the blank)

This tells us that *m* is also an even number. Why?

____

So, *m* = 2*m*' and *n* = 2*n*'.

Recall that, 1 = *my* and *s* = *ny*

So, we get that, 1 = 2*m*'*y* and *s* = _____

That is, 1 = $m' \times 2y$ and s = $n' \times 2$ __

So, _____is also a common measure for 1 and *s*. (Fill in the blanks)

Now, 2*y* is obviously greater than the greatest common measure, *y*.

This actually contradicts what we had started with.

We were led into this absurdity as a result of assuming that the length *s* may be expressed as a ratio of whole numbers. So our assumption must be wrong.

> Talk to the children about the technique of ‘proof by contradiction’ and they might have doubts about the process of proving.
> Spend some time on it. You might have to convince the students that there is indeed a contradiction. Also here you are proving that $\sqrt{2}$ is an irrational number. Ask the students to construct a proof for $\sqrt{3}$ is an irrational number or $\sqrt{5}$ is an irrational number. At this point you can also ask the students whether $\sqrt[3]{2}$ is an irrational number or not. The same proof will work to prove that $\sqrt[3]{2}$ is an irrational number.
>
> <u>Here we are a very interesting point, we actually have a new definition for irrational numbers: “The lengths of line segments which do not have a common measure with a line segment of length 1 unit are called irrational numbers.”</u>

So diagonal of square and its side never have a common measure or are always incommensurable.

This particular result actually had a huge impact on how people did mathematics then. Until then geometry and arithmetic were looked at as one subject but because the arithmetic then was not advanced enough to provide a way for expressing by numbers the ratio of two lengths that do not have a common measure, people started looking at geometry and arithmetic as different subjects.

This proof shows the existence of line segments whose lengths could not be expressed by then existing numbers . It is said that because of this, arithmetic and geometry went their separate ways for more than two thousand years until an improved arithmetic made it possible to express by numbers the ratio of two lengths that do not have a common measure.

Part 2:

In the coming tasks, we will geometrically construct line segments of different lengths which are irrational numbers.

Task 5:

Draw a right angle triangle such that two of its sides are of unit length. What can you say about the length of its hypotenuse?

> If the students take the unit as cm, the line segment of 1 cm is too small to work with. You can ask the students to take 2 cm as their 1 unit and work.
>
> Some students might suggest that length of the hypotenuse can be found by measuring the length with a scale. Try to encourage them to find out the length without actually measuring.
>
> By Pythagoras Theorem we get,
> Length of the hypotenuse = $\sqrt{\left((1)^2+(1)^2\right)}=\sqrt{2}$

Task 6:

Using the hypotenuse obtained in Task 5 as one leg and one leg with unit length draw a right angle and complete the triangle. What is the length of hypotenuse of the new triangle?

Task 7:

Continue this process for minimum 5 steps.

Task 8:

Draw a similar spiral starting with one of the sides of the triangle as 6 units and the other as 1 unit instead of both sides of 1 unit. Continue for a minimum 5 steps. What can you say about the length of the last line segment you constructed?

After the children have constructed the spiral ask them to observe the lengths of the square-roots constructed. You can bring it to their notice that as n increases, the difference between square-roots of *n* and *n*+ 1 decreases.

Task 9:

How will you construct line segments whose length is equal to the following numbers? Give justifications as to why your answers are correct.

1) $\sqrt{32}$

There are many ways to construct a line segment of length $\sqrt{32}$ units.

i) Draw a line segment of 5 units and construct a spiral till $\sqrt{32}$ (7 iterations)

ii) $\sqrt{32}=\sqrt{(16\times 2)}=4\sqrt{2}$ so $\sqrt{2}$ constructed 4 times is $\sqrt{32}$ (If you construct 4 copies of line segment of length $\sqrt{2}$ end to end then you get a line segment of length $\sqrt{32}$)

iii)Draw a right angle triangle with one side of 4 units and other also as 4 units, then the hypotenuse is $\sqrt{(4^2+4^2)}=\sqrt{(16+16)}=\sqrt{32}$ units

(There are more ways than the ones given above.)

Encourage the students to justify why their methods will give the correct answer.

2) $\sqrt{40}$

i) Draw a line segment of 6 units and construct a spiral till $\sqrt{40}$ (4 iterations)

ii) $\sqrt{40}=\sqrt{(4\times 10)}=2\sqrt{10}$ so $\sqrt{10}$ constructed 2 times is $\sqrt{40}$ (2 copies of line segment, $\sqrt{10}$)

iii) Draw a right angle triangle with one side of 6 units and other as 2 units, then the hypotenuse is $\sqrt{(6^2+2^2)}=\sqrt{(36+4)}=\sqrt{40}$ units

(There are more ways than the ones given above.)

Encourage the students to justify why their methods will give the correct answer.

3) $\sqrt{50}$

i) Draw a line segment of 7 units and construct a spiral till $\sqrt{50}$ (1 iteration)

ii) $\sqrt{50}=\sqrt{(2\times 25)}=5\sqrt{2}$ so $\sqrt{2}$ constructed 5 times is $\sqrt{50}$

(There are more ways than the ones given above.)

Encourage the students to justify why their methods will give the correct answer.

4) $\sqrt{63}$

i) Draw a line segment of 8 units and construct a right angle triangle such that this line segment is the hypotenuse and the other side is 1 unit. (In some sense inverse of the spiral)
ii) Draw a line segment of 7 units and construct a spiral till $\sqrt{63}$ (14 iterations)
iii) $\sqrt{63}=\sqrt{(9\times7)}=3\sqrt{7}$ so $\sqrt{7}$ constructed 3 times is $\sqrt{63}$
(There must be more ways than the ones given above.) Encourage the students to justify why their methods will give the correct answer.
At this point you can talk to the students that there are a lot of irrational numbers which cannot be drawn like $\sqrt[3]{2}$, $\sqrt[5]{4}$ or π.

Part 3

Task 10:

Draw a right angle triangle such that its two sides which are at right angles are $\sqrt{2}$ and $\sqrt{3}$. ($\sqrt{2}$ and $\sqrt{3}$ can be drawn using the techniques you figured out in the first part of this learning unit.) What is the length of its hypotenuse?

Task 11:

If you draw a right angle triangle such that the two right angle sides are $\sqrt{n}$ and $\sqrt{m}$, what is the length of the hypotenuse?

While doing addition of irrational numbers, a lot of students think that $\sqrt{n}+\sqrt{m}=\sqrt{(n+m)}$. Objective of this task is for the students' to generalize that $\sqrt{n}+\sqrt{m}>\sqrt{(n+m)}$ for all *n* and *m* integers. If we have a right angle triangle with sides $\sqrt{n}$ and $\sqrt{m}$. Then the hypotenuse is always $\sqrt{(n+m)}$. Using triangle inequality, we will get $\sqrt{n}+\sqrt{m}>\sqrt{(n+m)}$. You can also encourage the students to come with an algebraic proof for this.

Task 12:

Can you draw a right angle triangle whose all sides are integers? Draw at least two different right angle triangles having this property. What kind of numbers did you get as side-lengths?

Task 13:

Can you draw a right angle triangle such that the two right angle sides are integers and the hypotenuse is an irrational numbers? Draw at least two different right angle triangles having this property. What kind of numbers did you get as side-lengths?

Ask the students to show that the hypotenuse is indeed an irrational number. Ask them how they found out the length of the hypotenuse.

Task 14:

Can you draw a right angle triangle such that the hypotenuse is an integer and one of the other sides is also an integer and the third side is an irrational number. Draw at least two different right angle triangles having this property. What kind of numbers did you get?

Take the hypotenuse to be 6 units. Then the square of the hypotenuse is 36. Find a partition of 36 such that one of the numbers is a perfect square and the other is not, like 16 and 20.

Now, $36=16+20$

$6^2=4^2+\left(\sqrt{20}\right)^2$

So the sides of the right angle triangle are $4,\sqrt{20},6$

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.

The idea is to pick any perfect square, say n^2 then partition this number in such a way that one of the numbers is a perfect square and the other one is not. This is always possible for $n>1$.

$(n-1)^2$ and n^2 - $(n-1)^2$ is one such example for n > 3. Then n, n-1 and $\left(\sqrt{n^2-(n-1)^2}\right)$ are the sides of these triangles.

Task 15:

Can you draw a right angle triangle such that the two right angle sides are irrational numbers and the hypotenuse is an integer? Draw at least two different right angle triangles having this property. What kind of numbers did you get?

A lot of examples of such triangles can be found without actually constructing. For examples: Take the hypotenuse to be 6 units. Then the square of the hypotenuse is 36. Find a partition of 36 such that both the numbers are not perfect squares like 14 and 22.

Now, $36=14+22$

$6^2=\left(\sqrt{14}\right)^2+\left(\sqrt{22}\right)^2$

So the sides of the right angle triangle are $\sqrt{14},\sqrt{22},6$.

You can ask students to explore many such examples. Try to encourage them figure out a strategy to get such examples.

The idea is to pick any perfect square, say n^2 then partition this number in such a way that both the numbers are not perfect squares. This is always possible for $n>1$. n^2 - 2 and n^2 is one such example. Then n^2, $\left(\sqrt{n^2-2}\right)$ and $\sqrt{2}$ are the sides if these triangles.

Task 16:

Can you draw a right angle triangle such that the hypotenuse is an irrational number and one of the other sides is also an irrational number and the third side is an integer. Draw at least two different right angle triangle having this property. What kind of numbers did you get?

A lot of examples of such triangles can be found without actually constructing. For examples:
Take the hypotenuse to be $\sqrt{26}$ units. Then the square of the hypotenuse is 26. Find a partition of 26 such that one of the numbers is a perfect square and the other one is not like 16 and 10. Then the triangle the lengths of sides $\sqrt{16}$ and $\sqrt{10}$.
Now,

$$26=16+10$$

$$(\sqrt{26})^2=4^2+(\sqrt{10})^2$$

So the sides of the right angle triangle are $\sqrt{10},4,\sqrt{26}$.

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.

The idea is to pick any number which is not a perfect square then partition this number in such a way that one of the numbers is a perfect square and other one is not. This is always possible for $n>1$.

Task 17:

Can you draw a right angle triangle whose all sides are irrational numbers? Draw at least two different right angle triangles having this property. What kind of numbers did you get?

A lot of examples of such triangles can be found without actually constructing.
For example: Take the hypotenuse to be $\sqrt{26}$ units. Then the square of the hypotenuse is 26. Find a partition of 26 such that both the numbers are not perfect squares like 14 and 12.
Now, $26=14+12$

$$(\sqrt{26})^2=(\sqrt{14})^2+(\sqrt{12})^2$$

So the sides of the right angle triangle are $\sqrt{12},\sqrt{14},\sqrt{26}$.

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.

The idea is to pick any non-perfect square, say n then partition this number in such a way that both the numbers are not perfect squares. This is always possible for $n>1$.

References

The Dangerous Ratio (https://nrich.maths.org/2671)
To Infinity and Beyond: A Cultural History of the Infinite
Dantzig, Tobias (1954). Number, the Language of Science. New York, Free Press.