Exploring Patterns in Square Numbers

Time Required:
Minimum 3 periods of 40 minutes each

Type of LU:
Classroom

Summary
In this unit students explore patterns in square numbers. Students look for and find patterns in square numbers arranged in different ways. Some of these patterns may be familiar to them and they may come up with more patterns as well , with careful observation. The unit aims to introduce students to some of the basic practices and ways of thinking of mathematics as a discipline – for example, reasoning, conjecturing, proving, problem solving, representing, seeing and using connections, generalizing, abstracting, and communicating precisely.

Learning Objective
(i) Observing and coming up with patterns
(ii) Expressing a pattern algebraically
(iii) Extending and verifying a pattern
(iv) Visualizing and proving a pattern
(v) Understanding the difference between verification and proof, difference between theorem and conjecture.
(vi) Exploring relationships between statements
(For example – does one follow from another, is one a special case of another, is one a generalisation of another etc)

Materials Required
Graph paper, Printed number grids

Prerequisites
Square numbers, Algebraic Identities, Operations on Polynomials

Suggested Readings
https://betterexplained.com/articles/surprising-patterns-in-the-square-numbers-1-4-9-16/
https://nrich.maths.org/7117
https://nrich.maths.org/2280
For a better understanding of modular arithmetic:
https://nrich.maths.org/4350
https://betterexplained.com/articles/fun-with-modular-arithmetic/

Introduction

Task 1:

The following are the squares of numbers from 1 to 20.

Number	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Square	1	4	9	16	25	36	49	64	81	100	121	144	169	196	225	256	289	324	361	400

What patterns do you see?

Invite students to come up with as many patterns as they can from the above table. The following are likely to come up. If they do not, nudge them to notice these patterns.

1. The squares of all odd numbers are odd.
2. The squares of all even numbers are even.
3. The last digits follow the repeating pattern 1, 4, 9, 6, 5, 6, 9, 4, 1, 0
4. Symmetry in the first nine of the above digits about 5
5. The difference between successive square numbers is the sequence of odd numbers.

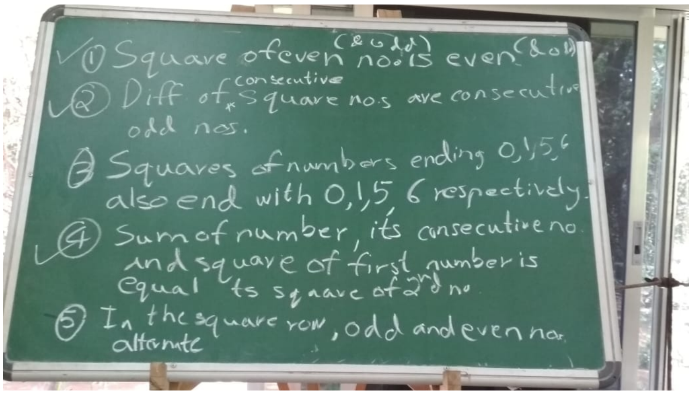
Students may observe many more patterns. Like

6. The square of a multiple of 10 always ends in 2 zeros.
7. For numbers 4 and greater, the number of digits in the square number is one more than that in the number. After giving them sufficient time to come up with patterns, call on them to read out their patterns and put them on the board.

These statements which the students come up with, can be called conjectures, as they are informed guesses which the students have made and it has not yet been established if they are always true or not. A conjecture that is always true is called a theorem.

Some of the patterns like (7) above may not be always true. Still write them on the board,instead of rejecting them outright. These can be picked up at a later stage and used to show how to prove and disprove conjectures.

Such conjectures that are not always true gives an opportunity to talk about how one can disprove a conjecture by coming up with one example where it will not be true. (Such examples are called counter examples.)



Some patterns that students came up with

Task 2:

Given below is a number chart. Shade the squares that have square numbers. The first few are done for you. What patterns do you notice?

I	II	III	IV	V	VI	VII	VIII		I	II	III	IV	V	VI	VII	VIII
1	2	3	4	5	6	7	8		209	210	211	212	213	214	215	216
9	10	11	12	13	14	15	16		217	218	219	220	221	222	223	224
17	18	19	20	21	22	23	24		225	226	227	228	229	230	231	232
25	26	27	28	29	30	31	32		233	234	235	236	237	238	239	240
33	34	35	36	37	38	39	40		241	242	243	244	245	246	247	248
41	42	43	44	45	46	47	48		249	250	251	252	253	254	255	256
49	50	51	52	53	54	55	56		257	258	259	260	261	262	263	264
57	58	59	60	61	62	63	64		265	266	267	268	269	270	271	272
65	66	67	68	69	70	71	72		273	274	275	276	277	278	279	280
73	74	75	76	77	78	79	80		281	282	283	284	285	286	287	288
81	82	83	84	85	86	87	88		289	290	291	292	293	294	295	296
89	90	91	92	93	94	95	96		297	298	299	300	301	302	303	304
97	98	99	100	101	102	103	104		305	306	307	308	309	310	311	312
105	106	107	108	109	110	111	112		313	314	315	316	317	318	319	320
113	114	115	116	117	118	119	120		321	322	323	324	325	326	327	328
121	122	123	124	125	126	127	128		329	330	331	332	333	334	335	336
129	130	131	132	133	134	135	136		337	338	339	340	341	342	343	344
137	138	139	140	141	142	143	144		345	346	347	348	349	350	351	352
145	146	147	148	149	150	151	152		353	354	355	356	357	358	359	360
153	154	155	156	157	158	159	160		361	362	363	364	365	366	367	368
161	162	163	164	165	166	167	168		369	370	371	372	373	374	375	376
169	170	171	172	173	174	175	176		377	378	379	380	381	382	383	384
177	178	179	180	181	182	183	184		385	386	387	388	389	390	391	392
185	186	187	188	189	190	191	192		393	394	395	396	397	398	399	400
193	194	195	196	197	198	199	200		401	402	403	404	405	406	407	408
201	202	203	204	205	206	207	208		409	410	411	412	413	414	415	416

As in the previous task, give sufficient time for students to study the table and come up with patterns. Write down all patterns that come up on the board. Let them draw a line joining 1,4,9,16, 25, ... the squares in order and see what pattern emerges.
It is obvious to see that the the shaded numbers appear only in columns I, IV and VIII. There is a regularity in the gaps between the shaded numbers, which may not be so obvious.
8. In column 1, the gaps increase as 1, 2, 3, 4, 5,...
9. In column IV, the gaps increase as 4, 8, 12, 16, ...
10 In column VIII, the gaps increase as 6, 10, 14, 18, ...

Write down on the board the patterns that students come up with in the form that they come up with. Nudge them to noticing the following patterns.

11. The squares of all odd numbers are in column I
12. The squares of even numbers are in columns VI or VIII
13. The squares of multiples of 4 are in column VIII.
14. The squares of even numbers that are not multiples of 4 are in column IV.

You may want to ask students to write out the conjectures that they come up with on chart papers and make posters that can be displayed in the classroom or Math Lab.

If the students are not coming up with patterns (8), (9) and (10) they can be ignored or let only the interested students pursue these patterns, They need not be imposed on all students. The proofs or statements related to them are marked with an asterisk mark (*) in the subsequent document but make sure that they notice patterns (11) – (14). The students may come up with many other patterns other than those listed here. Write them all on the board.

Once all the patterns that students have found are on the board, we could do the following.

A. Checking if the pattern extends to all numbers or verifying it. This we do by generating more examples and seeing if it is true. We could also look for counterexamples in cases where the statements are false.
B. Reframing the statements in a mathematically more appropriate fashion if necessary.
For eg: statement 11 above can be reframed as
“The squares of all odd numbers leave a remainder 1 when divided by 8”.(11a)
or as
“The square of all odd numbers are of the form $8n + 1$.”(11b)

C. Checking if there are interdependencies among the statements – for example does one follow from the other, Is one a special case of another etc. Example: Statement (1) follows from statement (11a) or (11b). These are stronger statements than statement (1)

D. Proving the conjectures that are verified to be true.

Task 3 is about further exploring, visualising and proving some of the statements (from (1) to (14)) listed above. These may be done in any order that suits your class.

You may want to prove some of the conjectures generated by students in Task 1 as well.

Together tasks 1 and 2 are intended to motivate students to come up with conjectures, verify them, come up with counter-examples or proofs as the case may be. The specific conjectures you decide to take up to examine in detail, or the order you choose to do it is entirely up to you and can be decided by the conjectures that students come up with in your class. The following tasks are arranged in one such order. It is not necessary that you should stick to this order or prove these same conjectures as long as the intention of these tasks as mentioned above are satisfied.

Also students may come up with multiple ways of proving the same result . Make sure to discuss as many of these as possible and compare them.

Task 3A1:

Predict in which column the following will appear without evaluating? Justify your answer. How do you know for sure?

a) 75^2 b) 96^2, c) 122^2, d) 100000^2 e) 12346^2

> This task is meant to give an opportunity for students to use the patterns that they have come up with, and thus realise the power of the patterns. 75 being an odd number, 75^2 will be in the first column and leaves a remainder 1 when divided by 8. (Using statement (11) and its reformulations)
>
> Similarly 12346 being an even number that is not divisible by 4, its square will be in column IV and leaves a remainder 4 when divided by 8. (using statement (14) and its reformulations). For even such large numbers, we are able to find out where in the grid their squares will appear and what remainder the square will leave when divided by 8, even without either knowing what the square is or dividing it by 8. Let students figure out the answer for the remaining problems. Ensure that they are able to articulate the reasoning as to why they give a particular answer.
>
> Similarly knowing that the square of all odd numbers will be in column I, enables us to say that the square of $(1234567)^2$ is not in column 5. In order to be able to say any of the above with certainty, we need to prove that that patterns are always true. Get students to prove as many patterns as possible from the ones that they have come up with. Instead of writing out the proof, you could nudge them to the proof by giving appropriate prompts.

Task 3A2:

Is it possible that 1234567^2 is in column 5? Why do you think so?

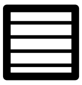
Task 3A3:

Prove at least 4 of the patterns that you came up with in Tasks 1 and 2.

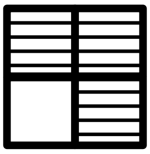
> This task is meant to give an opportunity for students to use the patterns that they have come up with, and thus realise the power of the patterns. 75 being an odd number, 75^2 will be in the first column and leaves a remainder 1 when divided by 8. (Using statement (11) and its reformulations)
>
> Similarly 12346 being an even number that is not divisible by 4, its square will be in column IV and leaves a remainder 4 when divided by 8. (using statement (14) and its reformulations). For even such large numbers, we are able to find out where in the grid their squares will appear and what remainder the square will leave when divided by 8, even without either knowing what the square is or dividing it by 8.
>
> Let students figure out the answer for the remaining problems. Ensure that they are able to articulate the reasoning as to why they give a particular answer.
>
> Similarly knowing that the square of all odd numbers will be in column I, enables us to say that the square of $(1234567)^2$ is not in column 5. In order to be able to say any of the above with certainty, we need to prove that that patterns are always true.
>
> Get students to prove as many patterns as possible from the ones that they have come up with. Instead of writing out the proof, you could nudge them to the proof by giving appropriate prompts.

Task 3B:

Here is a square.



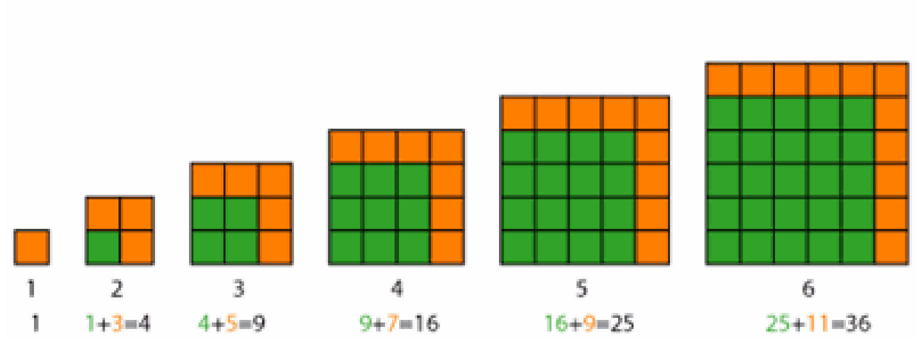
We can add three more squares to make it into a 2 x 2 square.



2

Build on this to make a 3 x 3 square. How many squares did you have to add?
Now use a 3 x 3 square to make a 4 x 4 square. How many squares did you have to add?

Continue the process for a few more steps. What patterns do you observe? Is it related to any of the patterns that you noticed earlier? Can you prove the pattern?

> This task is a visualisation of the fact that the difference between squares of consecutive numbers is an odd number.
>
> 
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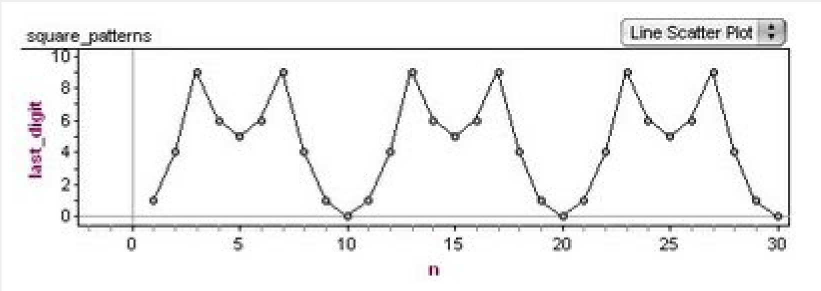
>
> Note that to build a 3 x 3 square on a 2 x 2 square, the outer edge of orange squares need to be added. Similarly to build higher order squares, one should keep on adding the outer edge.
> The outer edge that is to be added to a 3 x 3 square to make it a 4 x 4 square consists of 3 + 4 = 7 squares. (see the 4^{th} square above)
> Similarly, the outer edge that needs to be added to a 4 x 4 square to make it a 5 x 5 square consists of 4 + 5 = 9 squares.(5^{th} figure above)
> In general, the outer edge that needs to be added to a n x n square to make it an (n + 1) x (n + 1) square will have $n + n + 1 = 2n + 1$ squares, which is (n + 1)th odd number.
> This could be proved algebraically as well.
>
> The difference between two consecutive squares can be written as 0
> $(n + 1)^2 – n^2 = 2n + 1$
>
> Note that the result itself may be stated in many forms
>
> Like the “difference between the squares of two consecutive numbers is equal to the sum of those numbers”

Task 3C

Task 3C1 :

Plot the last digit of the square of a number against the number. What patterns do you see?

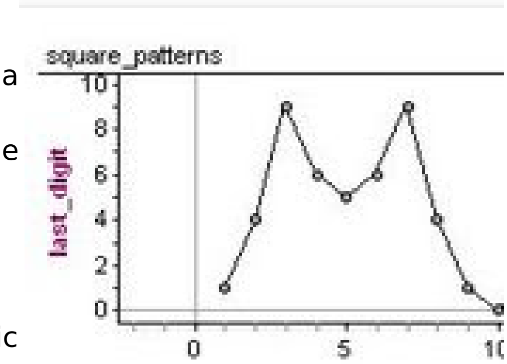
The students should get a graph like the following. Notice the repeating pattern and the symmetry about the lines at x = multiples of 5. The graph helps them see the repeating pattern and the symmetry.



Task 3C2

Notice that the pattern of last digits 1, 4, 9, 6, 5, 6, 9, 4, 1 0 repeats.
Can you prove that the pattern of last digits repeats?

The units digit of any number is the remainder obtained on dividing the number by 10.
Any number can be written in the form $10n + k$
$(10n + k)^2 = 100n^2 + 20nk + k^2$
The units digit of $100n^2 + 20nk$ is 0 as it is a multiple of 10.
So the units digit of $100n^2 + 20nk + k^2$ is the same as the last digit of k^2.
The units digit of k^2 and $(10n + k)^2$ are identical
Thus there is a repeating pattern.



*Also notice that the repeating unit is symmetric about multiples of 5. Which means that the last digit of $5n + k$ and $5n - k$ are the same. This can be proved as follows.
$(5n + k)^2 = 25n^2 + 10k + k^2$
$(5n - k)^2 = 25n^2 - 10k + k^2$
Or $(5n + k)^2 - (5n - k)^2 = 10k$
The difference between these two numbers is $10k$, which has a zero at the units digit and therefore they both have the same units digit.
Also the last digit of k^2 and $(10n - k)^2$ are identical.

Task 3C3

Instead of the grid in Task 1 if we were to have a 10 column grid like the following, and we shade the squares with perfect squares, can you guess in which columns these numbers would be? Are there columns where there would be no shaded numbers?
If you were to divide a perfect square by 10, what possible remainders would do you get?

I	II	III	IV	V	VI	VII	VIII	IX	X
1	2	3	4	5	6	7	8	9	10
11	12	13	14	15	16	17	18	19	20
21	22	23	24	25	26	27	28	29	30
31	32	33	34	35	36	37	38	39	40
41	42	43	44	45	46	47	48	49	50
51	52	53	54	55	56	57	58	59	60
61	62	63	64	65	66	67	68	69	70
71	72	73	74	75	76	77	78	79	80
81	82	83	84	85	86	87	88	89	90
91	92	93	94	95	96	97	98	99	100
101	102	103	104	105	106	107	108	109	110
111	112	113	114	115	116	117	118	119	120
121	122	123	124	125	126	127	128	129	130
131	132	133	134	135	136	137	138	139	140
141	142	143	144	145	146	147	148	149	150
151	152	153	154	155	156	157	158	159	160
161	162	163	164	165	166	167	168	169	170
171	172	173	174	175	176	177	178	179	180

> Help students make the connection between Task 2 and Task 3C. Task 2 was based on the remainders that a perfect square would leave when divided by 8, where as Task 3C was about the remainders that a perfect square would leave when divided by 10. Both of them are essentially similar. The remainder when divided by 10 happens to be the last digit as well. Just as we explored patterns in remainders when a perfect square is divided by 8 and 10, it is possible to explore patterns in remainders when a perfect square is divided by other numbers like 3, 4 as well. For example when a perfect square is divided by 3 one CANNOT get a remainder of 2. Some of the explorations below target such patterns. These can be taken up as homework assignments for interested students.

Task 3C4:

Guess what patterns would you see if the numbers were written in a 3 column grid. Where would the square numbers be?

I	II	III		I	II	III		I	II	III		I	II	III
1	2	3		61	62	63		121	122	123		181	182	183
4	5	6		64	65	66		124	125	126		184	185	186
7	8	9		67	68	69		127	128	129		187	188	189
10	11	12		70	71	72		130	131	132		190	191	192
13	14	15		73	74	75		133	134	135		193	194	195
16	17	18		76	77	78		136	137	138		196	197	198
19	20	21		79	80	81		139	140	141		199	200	201
22	23	24		82	83	84		142	143	144		202	203	204
25	26	27		85	86	87		145	146	147		205	206	207
28	29	30		88	89	90		148	149	150		208	209	210
31	32	33		91	92	93		151	152	153		211	212	213
34	35	36		94	95	96		154	155	156		214	215	216
37	38	39		97	98	99		157	158	159		217	218	219
40	41	42		100	101	102		160	161	162		220	221	222
43	44	45		103	104	105		163	164	165		223	224	225
46	47	48		106	107	108		166	167	168		226	227	228
49	50	51		109	110	111		169	170	171		229	230	231
52	53	54		112	113	114		172	173	174		232	233	234
55	56	57		115	116	117		175	176	177		235	236	237
58	59	60		118	119	120		178	179	180		238	239	240

Task 3C5:

Guess what patterns would you see if the numbers were written in a 4 column grid. Where would the square numbers be?

Task 3C6:

Can you make grids with other number of columns? Can you predict where the square numbers would be with such grids?

Task 4:

> As further exercises in reasoning and using some of the patterns above, students may be asked to answer questions like the following.

i) Without evaluating the square root can you figure out if the following are perfect squares? Also explain how you came to your decision.

a) 122 b) 356 c) 340 d) 12356 e) 425

ii) Give an example of a number that is a NOT perfect square and

a) whose units digit is 0, leaves a remainder 4 when divided by 8
b) whose units digit is 6, leaves a remainder 4 when divided by 8 and leaves a remainder 1 when divided by 3

Multiple reasons /arguments may be articulated, for example –

(a) The last digits of a perfect square follow the pattern 1,4,9,6, 5,6,9,4,1,0. The digit 2, the last digit does not belong to this pattern. Therefore it is not a perfect square. OR

(b) A perfect square leaves a remainder 1,4 or 0 when divided by 8. 122 leaves a remainder of 2 and therefore it is not a perfect square.

(c) 11^2 is 121, 12^2 is 144. So all numbers between these two numbers have to have their square roots between 11 and 12 and are therefore not perfect squares. Ensure that students articulate their reasons correctly and discuss as many arguments as possible in the class.

Notice that some of these arguments may not work for some numbers 356 has the last digit 6 and leaves a remainder 4 when divided by 8. So arguments of the form (a) and (b) will not work in this case. However an argument can be built up along the lines of (c). Other reasons like

(d) A perfect square leaves a remainder 1 or 0 when divided by 3, where as 356 leaves a remainder 2. Hence it is not a perfect square can also come up. In the case of 340 some students may come up with reasons like.
(e) A perfect square if it has the units digit 0, must have the hundreds digit 0 as well. This is not the case with 340 and therefore it is not a perfect square.
Encourage students to come up with as many arguments as they can. Also you may want to try this exercise with more numbers.

Also notice that the patterns here are stated as “The squares of all odd numbers are in column I”

or equivalently “If a number is the square of an odd number it is in column I”. This does not imply that “all numbers in column I are squares of odd numbers.” or equivalently “If a number is in column I it is the square of an odd number”

For example the number 425 is in column I, but not a perfect square. There are many more numbers i column I which are not perfect squares. The same holds about the other properties as well. Using the language of mathematics, last digit being one of 1,4,9,6,5 or 0, or leaving remainders 0, 1 or 4 when divided by 8 are necessary conditions for a number to be a square. They are not sufficient conditions.

> Use sufficient examples to help students appreciate this. By this argument, it is very much possible that a number satisfies all of these conditions and is still not a perfect square –
>
> Consider the number 1000:
>
> It has the last digit 0, digit in the hundreds place is also 0, leaves a remainder 0 when divided by 8, 1 when divided by 3 and yet is not a perfect square. To establish that a number is a perfect square, one has to find its square root and check – none of these conditions are sufficient.
> You may also ask them to come up other examples which satisfy a combination of these conditions, but the number is not a perfect square as in task ii above.

References:
https://betterexplained.com/articles/surprising-patterns-in-the-square-numbers-1-4-9-16/
https://nrich.maths.org/7117
https://nrich.maths.org/2280
For a better understanding of modular arithmetic:
https://nrich.maths.org/4350
https://betterexplained.com/articles/fun-with-modular-arithmetic/