Is Your Polygon Same as Mine?

Overview

In this learning unit, the students will explore minimum conditions needed to construct a unique triangle or a quadrilateral, even a polygon. The objective of this learning unit is for the students to realize themselves that conditions which enable drawing a unique polygon are the same as the conditions of congruency they study in their syllabus.

Minimum Time Required

Triangle activity (Task 1 to 5)– 2 sessions of 40 minutes each
Quadrilateral activity (Task 6)– 2 sessions of 40 minutes each
Extension to polygons (Task 7 to 9)– 1 session of 40 minutes

Type of Learning Unit: Classroom activity

Introduction

Have you ever wondered how would you describe a triangle that is in your mind to somebody on a phone? What do you really tell, do you tell sides or angles? And is it possible, that the person would get the exact same figure that you had in mind? Moreover, how can you do that by giving minimum information? Today we will try to answer these questions by investigating some examples, making observations, and verifying or refuting these observations.

Learning Objectives

(i) To establish a connection between congruence and constructions of unique triangles.
(ii) To find out the number of conditions necessary to ensure congruence of triangles, quadrilaterals, and polygons.
(iii) To understand why certain sets of conditions are not criteria for congruence.

Links to Curriculum

NCERT Mathematics Textbook Class 7; Chapter 7: Congruence of Triangle,
NCERT Mathematics Textbook Class 8; Chapter 4: Practical Geometry.

Prerequisites

- Use of geometric tools like scale and compass
- Basic understanding of construction of triangles and quadrilaterals

Materials required

Blank sheets, Pencils, Erasers, Compass-boxes (compass, set-squares, protractor, and scale), scissors

Task 1: Drawing Your Triangle

1. Draw a triangle of your choice on the given blank sheet of paper. Measure the sides and the angles of the triangle and name the triangle.

Before students begin working on this activity, make sure they are sitting in pairs.
Each task requires discussing with other students, and in the text, we often refer to these students in pairs as partners. Make sure they all have required materials or at least one set of scale and compass per bench.
This activity will give you a chance to see whether students can draw triangles, measure sides and angles, and also, whether they know how to label a triangle. See Figure 1, an example of student writing angle of a triangle on the side when we conducted this activity with grade 8 students.

2. Now see triangles drawn by your friends. Do you see anything interesting? What is it?

Keep the paper safe, on which you drew your triangle, we will be coming back to this triangle later in the activity.

Task 2a: Constructing a Triangle When One Side is Given

You know that only one side of the triangle is 6 cm. Now using this information draw a triangle on the given paper. Name your triangle.

> Students here might ask a question whether all the sides of the triangle are 6 cm, and a productive answer to this question is to say "NO". For the purpose of our activity, it is best not to get equilateral triangles. You can respond to them as, "One of the sides of the triangle is 6 cm, and not all". Said this, a student still might draw equilateral triangles, and we discuss later, how to handle such a situation.

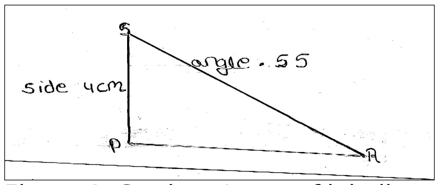

Figure 1: Students' way of labeling a triangle

Now study the triangle drawn by your partner.
1. Is your triangle same as your partner's?

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2. How did you compare these two triangles?

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3. You and your partner both were given one side, 6 cm. Did you both get exactly the same triangles? Why?

> Make sure that you collect some oral responses here. The students can give various criteria they used to compare the triangles like lengths of sides or measure of the angles. When you move around, check if there are pairs of students whose triangles are obviously different. Ask these students whether for such cases they need to measure the sides or angles or they can decide using some other ways.

Task 2b: Constructing a Triangle When One Angle is Given

You know that only one angle of the triangle is 55°. Now using this information draw a triangle on the given paper. Name your triangle.

> Students here might want to know the measures of the other angles. Use this opportunity to revise the properties of triangles and how given two angles one can always find the third angle.

Now study the triangle drawn by your partner.
1. Is your triangle same as your partner's?

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2. How did you compare these two triangles?

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3. You and your partner both were given one angle of 55°. Did you both get exactly the same triangles? Why?

Task 3: Construct a Triangle When Two Measures are Given

> Make three groups in the class, Group A, Group B, and Group C. Make sure that the students sitting beside each other – on one bench are part of the same group. Give group A the measure of two sides, group B measure of one side and one angle and group C measure of two angles.

Group A: Draw a triangle whose sides are 5 cm and 3 cm. Name your triangle.
Group B: Draw a triangle whose one side is 6 cm and one angle is 55^0. Name your triangle.
Group C: Draw a triangle whose two angles are 50^0 and 75^0. Name your triangle.
Now study the triangle drawn by your partner.

1. Is your triangle same as your partner's?

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2. How did you compare these two triangles?

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Group A:
You and your partner, both were given two sides, 5 cm, and 3 cm.
3. Did you both get exactly the same triangles? Why?

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Group B:
You and your partner, both were given one side, 6 cm and one angle 55^0.
4. Did you both get exactly the same triangles? Why?

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Group C:
You and your partner, both were given two angles, 50^0 and 75^0.
5. Did you both get exactly the same triangles? Why?

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> Students most probably will verify whether the two triangles are the same or not by measuring the sides and angles of their triangles. The teacher can also suggest superimposing (cutting the two triangles and placing on each other to see whether they overlap each other exactly) the two triangles, and see whether the triangles obtained are congruent to each other or not.

Task 4: Construct a Triangle When Three Measures are Given

> The class is already divided into 3 groups. Now divide each group into 2 sub-groups. A1 & A2, B1 & B2 and C1 & C2
>
> Make sure that the students sitting beside each other – on one bench are part of the same group. Ask all the groups to construct triangles as given in the instructions. Before they start ask them to predict which group would get a unique triangle, which would not. Remember A2 and B1 should get different triangles. You might have to have a common discussion if everybody's triangles are the same, especially in the group A2. Once they have found out that only groups A2 and B1 do not have unique triangles. Ask them if conditions given in A2 and C1 are different or the same and why there are different

results for them.

Group A1: Draw a triangle XYZ such that XY = 4 cm, YZ = 6 cm and XZ = 7 cm
Group A2: Draw a triangle ABC such that, AB = 5 cm, BC = 6 cm and ∠ACB = 45^0.
Group B1: Draw a triangle IJK such that ∠IJK = 40^o, ∠JKI = 65^o and ∠IKJ = 75^o.
Group B2: Draw a triangle STU such that ∠UST = 50^o, ST = 3 cm and ∠STU = 65^o.
Group C1: Draw a triangle EFG such that EF = 7 cm, FG = 9 cm and ∠GEF = 90^o
Group C2: Draw a triangle PQR such that PQ = 5 cm, ∠PQR = 50^o and QR = 4 cm

Now study the triangle drawn by your partner.
1. Is your triangle same as your partner's?

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2. How did you compare these two triangles?

Task 5: Minimum Conditions For the Construction of a Unique Triangle

1. If you want your friend/partner to construct exactly the same triangle like the one you drew in Task 1, what minimum information you will have to provide such that she/he will also construct the exact same triangle?

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2. If you think there are more than one ways to provide information such that a triangle, same as the one you drew in Task 1 could be constructed, please mention all those here.

Allow students to make all kinds of possible conjectures. One common conjecture the students came up with: give all three angles and all three sides. In response to this, you can remind them to come up with a minimum set of information. The six things they just mentioned is a lot of information. Again, giving three sides and two angles is the same as giving three sides and three angles (triangle's angle sum is 180^o). So ensure minimum as well as independent information is given.
Give them enough time. Insist on writing down their conjectures– in any language they prefer, using diagrams, using text, whatever helps them to convey. After 10 to12 minutes, collect their conjectures for further discussion.

Go through students' conjectures, and look for possible patterns. Based on our experience students came up with following conjectures:

- Two sides and two angles
- All three sides (SSS)
- One side and two angles (ASA, AAS)
- Two sides one angle (SSA, SAS)
- All three angles (AAA)

Lead a discussion here, on which of these will not work. Students will soon figure out that the number of minimum conditions required is three; discuss to find out which three. For some conjectures such as SSA, you might have to be ready with examples. See one such example in figure 2. You can have counter-examples for AAA.

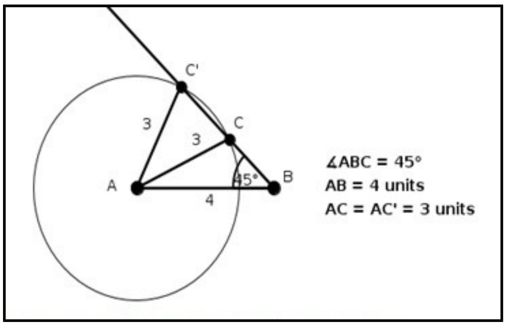

Figure 2: Refuting SSA using actual example

> Students will feel more convinced about SSS, and you can use this opportunity to ask them why they are so confident about it. Conclude this activity by talking about congruency tests that work, and elicit from students why would those work.

Task 6: Constructing a Quadrilateral

1. Now that you all know how to make a congruent triangle, let us figure out how to make a congruent quadrilateral. So if the minimum conditions for making a congruent triangle are three, what should be enough for a quadrilateral?

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> If the students' answer is four, ask them why they feel is four and which four. Ask them if you have information about all the four sides can you get a unique quadrilateral. Many students might draw squares first, but when prompted to get different quadrilaterals, they will get multiple rhombuses.

Now given that all the sides of a quadrilateral are 3 cm, think about all the different quadrilaterals that you can draw. Make the constructions on the given blank sheet.

2. Did you or your partner get different quadrilaterals?

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3. So if only sides are given, is it always possible to get different quadrilaterals? How do you know?

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4. Imagine that you have to write to your friend about a quadrilateral. Now think of minimum information that you can send him/her, such that he/she gets the exact same quadrilateral as the one you had in your mind. What information you will send?

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> Remember in case of quadrilaterals, you also have the diagonals so 4 sides, 4 angles, and 2 diagonals, a total of 10 measures form the maximum information. Give students enough time to come up with combinations of information. As we ruled out 4 as minimum information, let us stick to 5 minimum measures. Collect students' conjectures and lead the discussion for how it would work or not work. Some examples we received while working with Class 8 learners are as follows:
>
> - 4 sides and 1 diagonal
> - 2 adjacent sides and 3 angles
> - 3 sides and 2 included angles
>
> Another reasoning about why we need 5 pieces of information can be something like this, "to construct a quadrilateral one needs to fix four points or vertices, by now we know that to fix three points we need three conditions (triangle). Now we need to fix the fourth point. The information we have is in terms of either an angle or a length. An angle gives a straight line and a length gives us a circle. So it is clear that one condition is not enough to fix the fourth point so we will need at least five conditions.
>
> Students can actually construct these examples and see whether they get a pair of congruent triangles. However, a general strategy to understand is to see how two congruent triangles when we join them will give rise to a quadrilateral. This understanding can be used to deduce the congruency conditions for quadrilateral. Joining two triangles first reduces the information needed from 6 to 5 conditions as one side overlaps.

Check whether what you suggested as minimum information really works. Try drawing different quadrilaterals for the information you said you would give to your friend in the question above.

5. Think about why that set of information will lead to congruent or non-congruent quadrilaterals.

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6. List the conditions that worked for constructing a unique quadrilateral.

Some standard conditions that will give congruent quadrilaterals are given below for your reference.

- All four sides and the length of one of the diagonals
- All four sides and one of the angles
- Three adjacent sides and two included angles within those sides
- Three angles and two included sides within those angles

Task 7: Some Special Triangles and Quadrilaterals

We have found out the minimum information needed to draw congruent triangles and congruent quadrilaterals but let us look at some special triangles and quadrilaterals and find out what kind of minimum information we need to construct them.

1. How many conditions do you need to construct congruent equilateral triangles?

The students will find out that only the length of the equilateral triangle is enough to draw a congruent triangle. But in reality, by giving the length of the side of an equilateral triangle you are giving all the 6 pieces of information (3 sides and 3 angles). Do underline this while discussing this task.

2. How many pieces of information do you need to construct congruent squares?

The students will find out that only the length of the side is enough to draw a congruent square. But in reality, by saying that it is a square and giving the length of the side of a square you are giving all the 8 pieces of information (4 sides and 4 angles[We know that all angles are 90^o]). Do underline this while discussing this task.

3. How many pieces of information do you need to construct congruent rectangles?

The students will find out that in the case of rectangles, you need lengths of two adjacent sides. But in reality, by giving the lengths of two adjacent sides of a rectangle you are giving all the 8 conditions (4 sides: `opposite sides are equal' and 4 angles: `all angles are 90^o'). Do underline this while discussing this task.

4. How many pieces of information do you need to construct congruent rhombuses?

The students will find out that in the case of rhombus, you need the lengths of one side and one angle. But in reality, by giving the length of one side and measure of one angle of a rhombus you are giving all the 8 conditions (4 sides: 'all sides are equal' and 4 angles: ‘adjacent angles are complimentary and opposite angles are equal’). Do underline this while discussing this task.

5. How many pieces of information do you need to construct congruent parallelograms?

In the case of parallelograms, you need lengths of two adjacent sides and the including angle. But in reality by giving the lengths of two adjacent sides of a parallelogram and the including angle you are giving all the 8 conditions (4 sides:`opposite sides are equal’ and 4 angles: `We know that opposite angles are equal and adjacent angles are complimentary’). Do underline this while discussing this task.

6. How many pieces of information do you need to construct congruent trapeziums?

In the case of trapeziums, you need length of one non-parallel side, length of the base, and measure the two bases angles. Basically, by saying that the quadrilateral is a trapezium, you can construct the other parallel side from the given information.

If you look at all the tasks together, you will notice that for squares one needs only one piece of information explicitly, for rectangles and rhombus it will be two pieces of information, for parallelogram it is 3 and for trapeziums it is 4. If we recall, squares are special cases of rectangle or rhombuses, rectangles or rhombuses are special cases of parallelograms and parallelograms are special cases of trapezium. On the other hand, while constructing squares one needs only one explicit piece of information, for rectangles or rhombuses it is two for parallelogram it is 3 and for trapezium it is 4.

So as you construct more and more general quadrilaterals, you will need more pieces of information to construct them uniquely till you reach 5 pieces of information.

Task 8: Constructing a Pentagon

1. Now that you all know how to make a congruent triangle or a congruent quadrilateral, let us figure out how to make a congruent pentagon. So if the minimum conditions for making a congruent triangle are three, and that for a quadrilateral are 5, what do you think the number of minimum conditions needed to construct a unique pentagon is?

Mostly the students will answer it as 7 looking at the pattern. Though the answer is correct probe the students to find some 7 conditions such that they give a unique pentagon.

2. Imagine that you have to write to your friend about a pentagon. Now think of minimum information that you can send him/her, such that he/she gets the exact same pentagon as the one you had in your mind. What information you will send?

Like in case of quadrilaterals for pentagons also there will be diagonals so 5 sides, 5 angles, and 5 diagonals, a total of 15 measures form the maximum information. Give students enough time to come up with combinations of information. Collect students' conjectures and lead the discussion for how it would work or not work.

Check whether what you suggested as minimum information really works. Try drawing different pentagons for the information you said you would give to your friend in the question above.

3. Think about why that set of information will lead to congruent or non-congruent pentagons.

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4. List the conditions that worked for making a unique pentagon.

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> Some standard conditions that will give congruent quadrilaterals are given below for your reference.
> • All five sides and two of the diagonals
> • All four sides and three included angles
> • Four angles and their included sides.

Task 9: Finding the Number of Conditions to Construct Congruent Polygon

Now that you know the minimum conditions needed for making a congruent triangle and a congruent quadrilateral, let us explore how many conditions are needed for constructing a congruent hexagon, or a congruent heptagon.

Make some guesses, and make constructions on the given sheets of paper, record your guesses in the given table below.

Table 1

Number of sides in the polygon	Name of the polygon	Minimum conditions required for constructing a congruent polygon
3	Triangle	3
4	Quadrilateral	5
5	Pentagon	_______
6	Hexagon	_______
7	Heptagon	_______
8	Octagon	_______

> Some students might complete the table by observing patterns. Some students might struggle with the table. Give ample time to the students and then collect responses to complete the table on the board.

Proving Our Understanding

Let us find out how can we prove which guesses are right and which ones are wrong.

Draw a quadrilateral.

Draw the first triangle inside the quadrilateral.

See *Figure 3* here.

(Here we have drawn two different types of quadrilaterals)

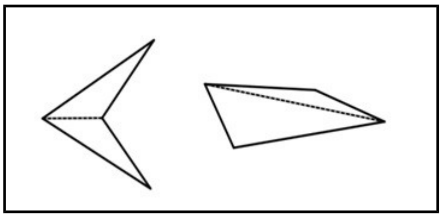

Figure 3: Trigulating Quadrilaterals

1. How many such triangles can be made inside each of these quadrilaterals?

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Yes, 2 such triangles can be made. We know that for constructing a congruent triangle we need 3 minimum conditions.
So in this case, to construct the first triangle we needed three minimum conditions. For the next triangle, we need three more, but as one side overlaps, you need only two conditions to construct triangle congruent to the second triangle. Also we know that to fix one point we need at least two conditions. Hence these 5 conditions are the minimum pieces of information needed to construct a quadrilateral.

This also reconfirms with our understanding of minimum conditions needed for constructing a congruent quadrilateral.
What will happen if we do the same for a pentagon?

Let us draw a pentagon and see how many triangles can be made inside a pentagon.

For the first triangle we need three conditions, for the second triangle we will need another 3 but then one side overlaps so we need only two. Similarly, for the third triangle, we need two more conditions.

So you can see that whenever you add a triangle, you add two conditions. So the minimum conditions necessary for constructing a congruent pentagon, are 7 (3 + 2 + 2). But these five conditions give us a unique pentagon, hence the minimum pieces of information needed to construct a unique pentagon is 7.

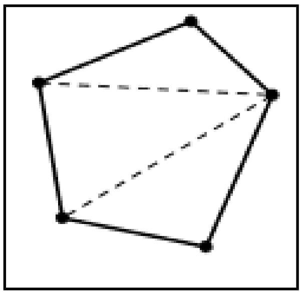
Figure 4

Let us try to figure this out for hexagons, heptagons, and octagons.

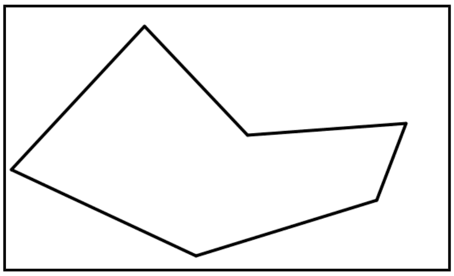
Figure 5

2. How many triangles in a hexagon? ___________________

3. What is the minimum number of conditions needed to construct unique hexagons? ___________________

4. Why?

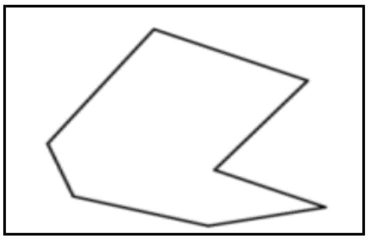
Figure 6

5. How many triangles in a heptagon? ___________________
6. What is the minimum number of conditions needed to construct unique heptagon? ___________________
7. Why?

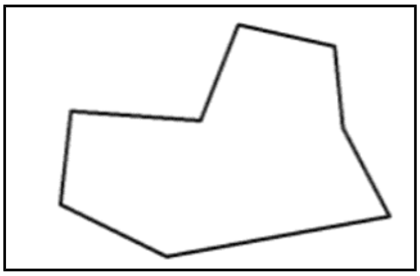
Figure 7

8. How many triangles in an octagon? ___________________
9. What is the minimum number of conditions needed to construct congruent octagon? ___________________

10. Why?

It is interesting to see what are the minimum pieces of information needed to construct a unique polygon. Let us start with a polygon with n sides. To fix the first three vertices, we need 3 conditions. For each of the *n* - 3 remaining vertices, we need at least two pieces of information or conditions hence we need at least 2(*n* - 2) + 3 = 2*n* – 3 conditions. But the process of dividing the polygon into *n* – 2 triangles tells us that 2n -3 conditions are sufficient to construct a unique polygon. So for a polygon, the number of minimum conditions needed to construct is 2n -3.

Suggested Readings

1. Simple argument about the number of minimum conditions needed to construct congruent polygons- https://www.mathopenref.com/congruentpolygonstests.html
2. A simulation to check congruent polygons by super-positioning- https://www.mathopenref.com/congruentpolygons.html
3. Interesting examples of congruence- https://www.andrews.edu/~calkins/math/webtexts/geom07.htm
4. Detailed proof of why the number of minimum conditions needed to construct congruent n-gons is 2n-3- http://www.amesa.org.za/amesal_n21_a12.pdf

References

- http://www.amesa.org.za/amesal_n21_a12.pdf
- https://www.mathopenref.com/congruentpolygons.html
- NCERT Mathematics textbook Class 7, Chapter 7; Class 8, Chapter 4