Counting Areas

Overview

In this unit, students are led through a guided discovery of the Pick's Theorem and its proof. It invites them to consider the relationship between the area of special cases of grid polygons[1] and see if it can be generalised to any grid polygon or how it should be modified so as to be applicable to the general case. Proofs for special cases are considered and directions given for a more general case.

Learning Objectives

To provide opportunities for students to engage in practices of mathematics such as
observe patterns,
formulate conjectures,
modify or refine them based on additional information,
generate examples to verify or refute conjectures,
prove conjectures,
consider special cases,
generalise them etc.

Material Required

Worksheet/Grid Paper

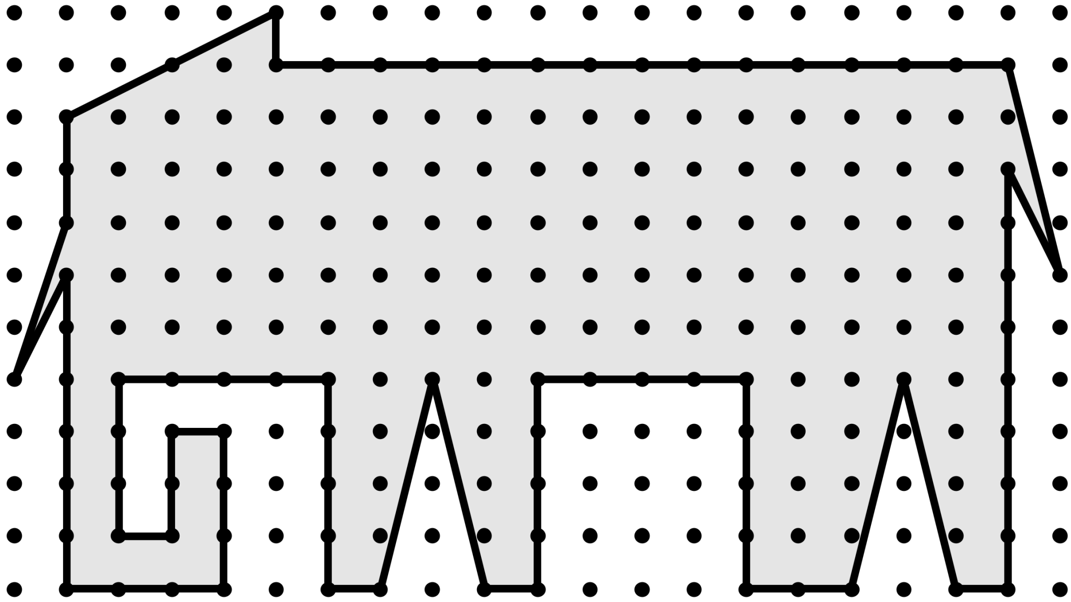
Time Required

3 sessions of 40 minutes each

Task 1:

King Bahubali loved elephants so much that he kept a herd of them. In fact, he planted his coconut garden in such a way that it looked like an elephant when viewed from his terrace! But the elephants would walk around the garden and destroy it. So the king put a fence around the garden to keep the elephants away as seen in the figure given below. The trees were planted on a square grid, with one tree at each grid point, so as to provide sufficient space for each tree. If the king's grounds were 20 units long and 11 units wide, can you find the area available for the elephants to roam, by just counting the coconut trees?
If you can't figure out, go ahead with the remaining tasks, and you will be able to do this at the end of the tasks!!

1 Grid polygons: Polygons whose vertices are points on a square grid

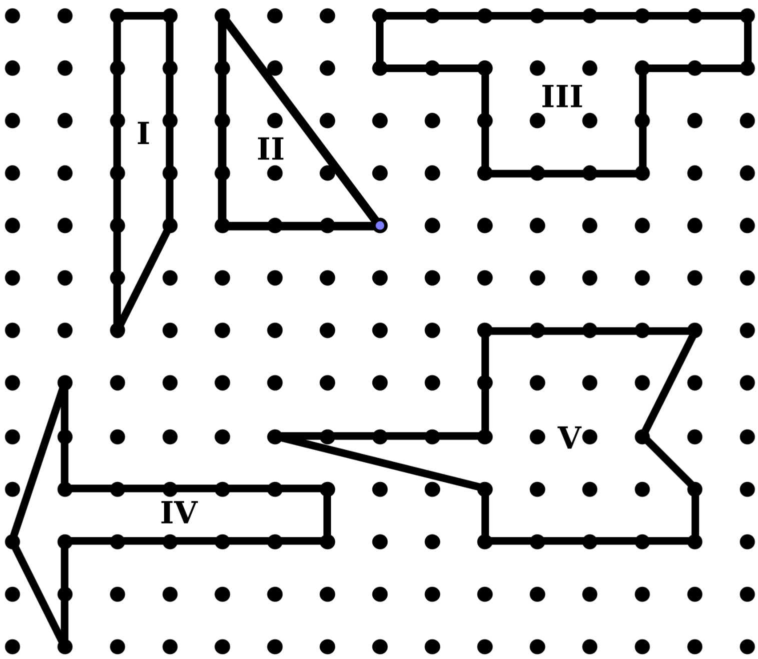


This task is meant as a motivation for the unit. Students are not expected to solve this at this point. They can come to the solution of this later, after engaging with the first few tasks of the learning unit.

Task 2:

Given below are some figures. Find the area of each and complete the given table.

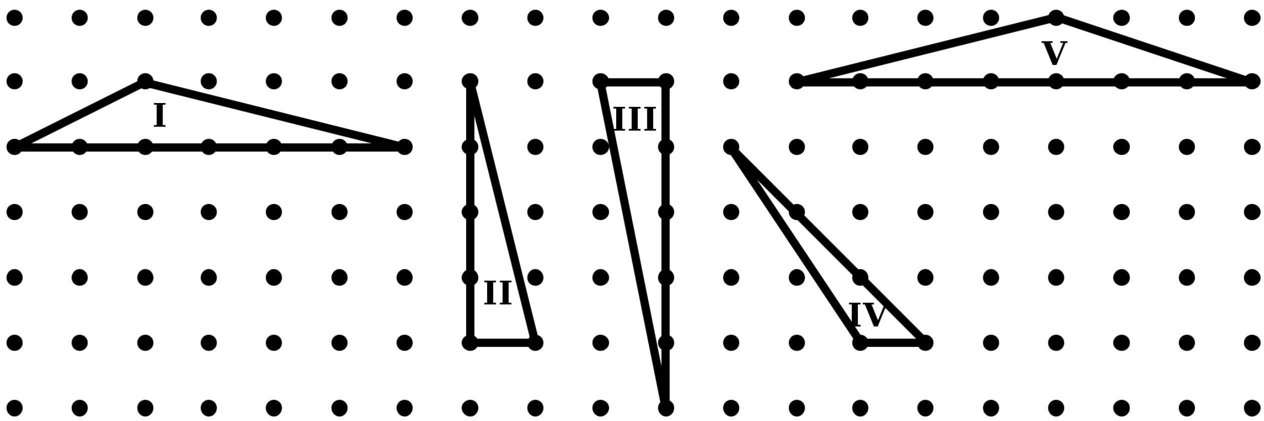
Figure	Area in Sq Units
I	
II	
III	
IV	
V	



Observe how students are finding the area of the figures. Encourage them to use multiple ways of finding areas – say counting grid squares, breaking up the figure into figures whose areas can be easily found out, or looking for ‘part-squares’ which add up to a complete grid-square etc.

Task 3

a) Find the area of the following triangles.



Also count the number of grid-points on the boundary of each triangle, and fill the table below.

Triangle	Area in Square Units	Number of grid-points on the boundary (B)
I		
II		
III		
IV		
V		

b) Do you see any relation between the area of the triangle and the number of grid-points on its boundary?
c) Does the same relation hold for figures I to V in Task 2? If not, for which ones does the relation hold?
I) The relation holds for figures ________________. (Write the number of the figure.)

ii) The relation does not hold for figures _______________. (---- ditto -----)

> The area of the triangles = $\frac{B}{2}$ - 1, where B is the number of grid-points on the boundary. The relation holds for those figures that do not have interior grid-points. The relation holds for figures I and IV of Task 2.
> The table in Task 2 could be extended and appropriate columns added, to find this out.

Task 4

a) In Task 3c), how are these figures in i) different from the figures in ii) ? What property distinguishes figures in i) from figures in ii)?

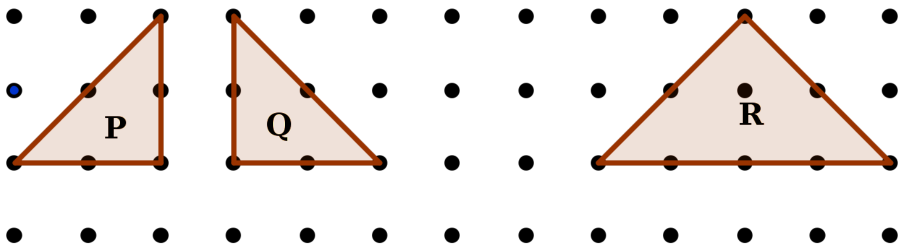
b) How would you modify the relation in Task 3b) such that it holds for all figures?

In Task 4a) the distinguishing property is that figures I and IV do not have grid-points inside them, where as the other figures have.

Observe what properties students come up with and if it is verifiable that all figures that have this property satisfies the relation, area = $\frac{B}{2}$ - 1. If not provide counterexamples of figures which do not satisfy the relation. This can be done by other students as well.

Provide sufficient time for students to engage with task 4 b). In case they are not able to come up with a modified relation, you may want to provide hints such as the following.

1) In the following figure, Triangles P and Q, both of which have no grid-points in their interior can be put together to make triangle R, which has one grid-point in the interior.

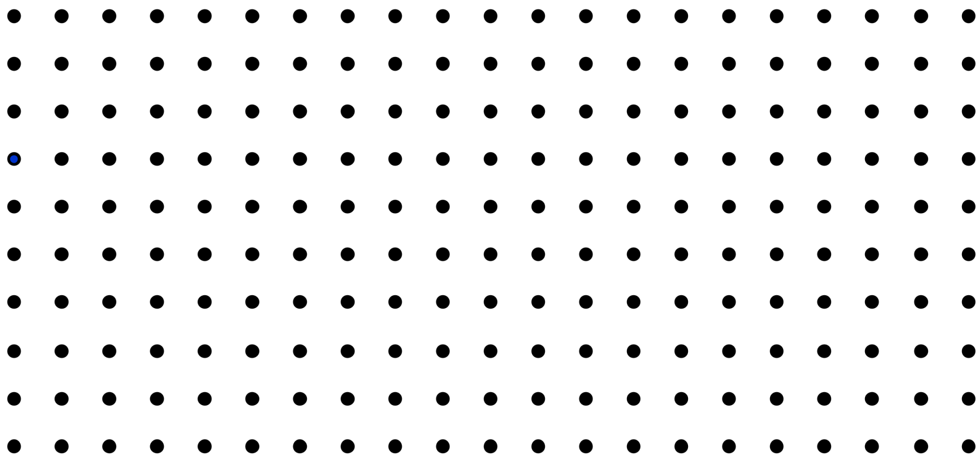


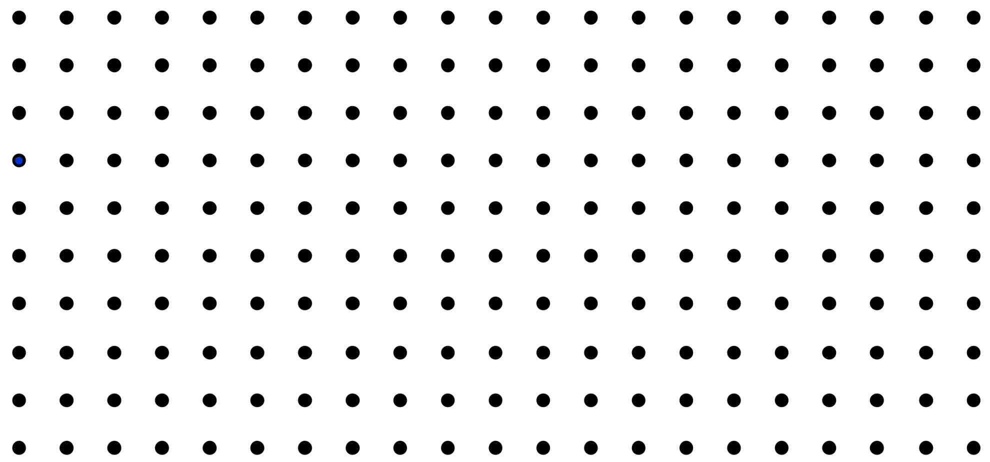
Which points on the boundary of triangles P and Q and on the boundary of R? Do some points on the boundary become interior grid-points now? Do some grid-points coincide when the triangles are put together?

2) Let students tabulate the number of grid-points in the interior, on the boundary and the difference between the area and the expression $\frac{B}{2}$ - 1. It can be seen that Area = I + $\frac{B}{2}$ - 1. where I is the number of grid-points in the interior.

Task 5

Draw five more figures on the grid provided below and check if the relation holds for these figures as well. Are you sure that it will hold for any figure that you may draw? What are the properties common to the figures for which this relation holds?





The relation holds for figures that have vertices on the grid points straight boundaries. That is the relation holds for grid-polygons.

Task 6

a) For a square of side *m* units

The number of grid-points in the interior (I) is ____________

The number of grid-points on the boundary (B)is ____________.

$I + \frac{B}{2} - 1 =$ ____________.

How is the expression $I + \frac{B}{2} - 1$ related to the area of the square?

b) For a rectangle of length *l* units and breadth *b* units

The number of grid-points in the interior (I) = _______________

The number of grid-points on the boundary (B) = ____________

$I + \frac{B}{2} - 1 =$ ____________

How is the expression $I + \frac{B}{2} - 1$ related to the area of the rectangle?

For a square of side *m* units,

The number of grid-points in the interior (I) is $(m - 1)^2$

The number of grid-points on the boundary (B)is $4m$.

$I + \frac{B}{2} - 1 = m^2$, the area of the square.

This can also be used as a visual proof of the identity $(m – 1)^2 + 4m = (m + 1)^2$

In the case of the rectangle,

The number of grid-points in the interior (I) is $(l – 1)(b - 1)$

The number of grid-points on the boundary (B) is $2(l + b)$.

$I + \frac{B}{2} - 1 = lb$, the area of the square.

Some students may need to consider a few specific cases before they arrive at general expressions for the number of grid-points in the interior and boundary of a general square and rectangle.

For a figure Q, with I grid-points in its interior and B grid-points in its boundary, let us call

$$I + \frac{B}{2} - 1 \text{ as } \textbf{Pick(Q).}$$

Then for a rectangle and square, we saw that

Pick(Q) = Area(Q)

This is called Pick's Theorem.

Thus we proved Pick's theorem for a restricted class of figures namely squares and rectangles. We now will go on to see if Picks Theorem is true for all figures on the grid paper. But before that, let us go back to Task 1!

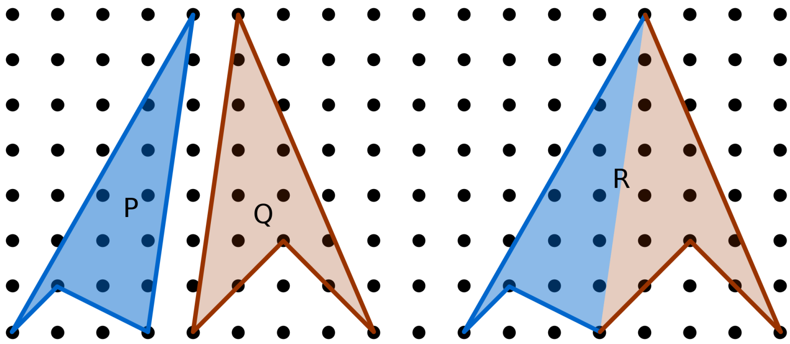
c) Solve Task 1.

Students can now use Pick's Theorem to find the area covered by the coconut trees by counting trees and subtract it from the area of the grounds (20 x 11 square units) to find the area where elephants can roam.

The following tasks give some hints to prove the Pick's Theorem for a general grid-polygon. This can be taken up with interested students depending on the availability of time.

Task 7

If we put together two figures, say figure P and figure Q, in such a way that they share a boundary, to form a compound figure R, the Area of (R) = Area (P) + Area (Q)



Let I_P and I_Q be the number of grid-points in the interior of P and Q respectively and B_P and B_Q be the number of grid points in the boundary of P and Q respectively.

Let is say that Pick(P) = $I_P + (B_P / 2) - 1$

Pick (Q) = $I_Q + (B_Q /2) - 1$

Since Pick (P) and Pick (Q) are the areas of P and Q respectively, and the areas of P and Q add up to give area of R, we would expect that Pick (P) and Pick (Q) would add up to give Pick (R) as well.

Now Pick (R) = $I_R + (B_R / 2) - 1$

where I_R and B_R are the number of grid-points in the interior and boundary of R.

Now, how are the number of grid-points in the interior of R related to those in the interior of P and Q?

Now how are the number of grid-points in the boundary of R related to those in the boundary of P and Q?

a) Can you come up with an expression for I_R and B_R in terms of I_P, I_Q, B_P and B_Q?

I_R = _______________ B_R = _________________

b) Substitute these in the expression for Pick (R) and verify that Pick (R) = Pick (P) + Pick (Q)

If c is the number of grid points on the common boundary,

$I_C = I_p + I_Q + c - 2$
$B_C = B_P + B_Q - 2c + 2$

Think, how these expressions come, why is there a '+ 2' and '-2'?

Therefore Pick (C) = $I_p + I_Q + c - 2 + ½ (B_P + B_Q - 2c + 2) - 1$
= $[I_P + (B_P/2) - 1] + [I_Q + (B_Q/2) - 1]$
= Pick (P) + Pick (Q)

Task 8

Use the result in Task 7 to prove Pick's Theorem for more general figures. {Hint: Start with right triangles, then move to any triangle and then onto figures that can be divided into triangles.}

What properties should the figure have for Pick's Theorem to hold?

An outline of the proof can be found here
https://nrich.maths.org/5441

or here
https://en.wikipedia.org/wiki/Pick%27s_theorem

In this unit in order to motivate the theorem, we have considered triangles which have no interior points and came up with the relation for this special case and then generalised it to other grid-polygons. In proving the theorem also, we have proved it for the special cases of a rectangle and a square and provided hints for the general proof. We started with a special case and moved to increasingly general cases.

This is just one of the ways of motivating the theorem and proof. You may want to consider other ways like

*Motivating the theorem itself through special polygons - rectangle and square

* Drawing grid-polygons with increasing number of interior points – starting from say 0, through 1, 2, 3.. and observing the relation between the area and the number of grid-points on the boundary and interior

* Observe the relation between area and the number of grid-points on the boundary and

interior of general grid-polygons

* or any other track that you may find comfortable.

The key idea is to have students observe and find patterns, come up with examples to verify or refute this conjecture and go on to prove it.

It is also a good idea to think of extending the task – For example, some of the questions that could be explored here are
* Would the theorem still hold if there are 'holes' in the polygon? How would one need to modify the theorem (if possible) to accommodate this case?

* Would the theorem still hold if some of the boundaries of the figure are curved? How would one need to modify the theorem (if possible) to accommodate this case?

* In case there are some curved boundaries, would it be possible to 'cover' the figure with a grid-polygon and thus come up with upper and lower bounds for its area?

Notice that we are considering more and more general cases of figures on a grid and exploring if the theorem holds for these figures. This is also an aspect of mathematics that could be conveyed to students through this unit. Invite students to propose variations and ask their own questions to extend the task. Even if the solutions to all extensions proposed are not found out, the exercise of thinking through these variations may itself be valuable.

References

https://kurser.math.su.se/pluginfile.php/15491/mod_resource/content/1/picks.pdf
http://www.geometer.org/mathcircles/pick.pdf
Ian Stewart (1992) - Another Fine Math You've Got me into , Dover Publications