Maps, Time-Tables and Radio Stations: What is common in all?
-Mathematical Modeling

The Learning objective

- To introduce mathematical modeling and creative problem solving
- To learn to represent a problem mathematically using graphs and the concepts of paths and colouring

Materials required:

Worksheets, Pens, Pencils, Erasers, Color Boxes, Board, Projector

Time required: 3hr or 2 sessions of 90min.

Recall:

- Have you already completed the Koenigsberg Bridge Learning?

  It is a good idea to conduct the Koenigsberg Bridge Learning Unit before conducting this unit
- Are you comfortable with basic definitions of vertex and edges?

Task 1 : Realistic problem

The king and his kingdom

Once upon a time, there was a kingdom ruled by a king who had three daughters and two sons. It was his wish that upon his death, this kingdom should be divided into five regions, one region for each child, such that everyone is everyone's neighbour, that is each region would have a common boundary with every other regions.

What do you think, can this be done? Think about and discuss the solution with your partner.

_____

_____

_____

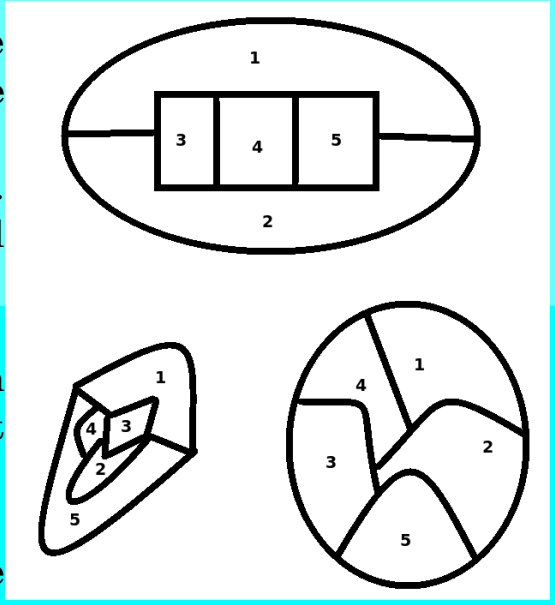
Give the students 5-7 minutes time to think about it. It may seem difficult when they start but once they start thinking, they will come up with many answers. Because, students will get a sense of why cannot they divide a region in 5 parts which are adjacent to each other.

Once they understand the problem they themselves come up with the idea of dividing regions in different ways.

Some of the questions students may ask:

- Can I draw any shape of the kingdom? - Yes, you can.
- What if they all share one point as a common boundaries, will it be considered? - A point is not considered as a boundary.
- Can we draw a scattered/disjoint kingdom? - A scattered disjoint kingdom cannot be considered as one kingdom.
- Is it necessary that all the regions will have equal areas? - The only condition is that they must share boundaries with each other. So it is not necessary that they have equal areas.

You might write down some of the students' responses on black-board. Some students might come up with the figures similar to the figures given below. These figures illustrates some unsuccessful attempts to satisfy the king's wishes.



In the first figure, kingdom 3 and kingdom 5 are not neighbours. That is they do not have a some common boundary.

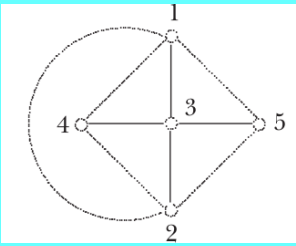
Similar problems can be notices in the other two figures.

Please note that the three figures given above are only examples and students can come up with many more examples.

Think about how you can simplify your figure. Remove the details which not required to solve the problem? Draw a simplified version of your figure and discuss with your partner how your picture/diagram still represents the problem given.

One of the ways to simplify the problem is the following: Recall figure 1, let each region be represented by points. So you have 1, 2, 3, 4 and 5 points. Join the points whose corresponding regions are neighbours.

You will get something like this.



This particular kind of representation is called a graph.

A graph is a collection of points (called vertices) and lines joining the vertices (called edges).

In this particular case, two vertices would have an edge between them if they are neighbours.

For this problem, in order to have a solution to the king's wishes, the resulting graph must have five vertices, each two joined by an edge.

This has been proved that this is impossible.
https://nrich.maths.org/6291
http://mathworld.wolfram.com/Four-ColorTheorem.html

Task 2 : Colouring a Map

Have you ever saw a coloured map of India? What have you noticed in it?
Students might response states, districts, sea and so on.
If you see any map in the atlas or on the internet (e.g. https://www.mapsofindia.com/), you will notice that in every map, regions which are next to each other; they share a boundary are always colored in different colours so that the regions next to each other can be distinguished. Let us try to explore this idea.
Q. What is the minimum number of colours you have observed in any map?_____

Colouring 1
We are giving you a map, can you colour it in such a way that every marked region can be clearly distinguished from its neighbours. Try to use minimum number of colours as possible.

Try to estimate the number of colours needed before actually starting to colour and then do actual colouring.

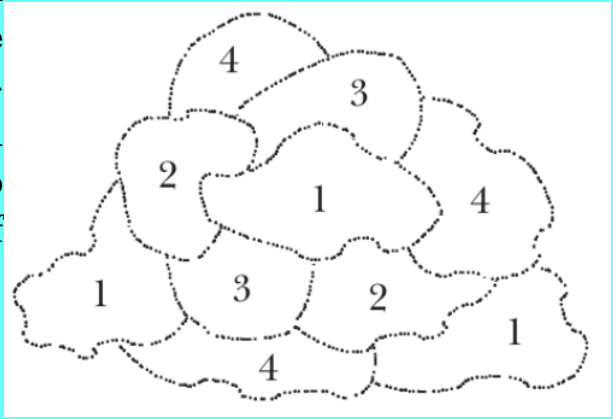
Q. What is the minimum number of colours you needed? _____

Colouring 2

Try to estimate the number of colours needed before actually starting to colour and then do actual colouring.

Q. How many minimum colours you actually need to colour the given map? _____

Make a table on the black board with two columns. In the first column put the number of colours used (put numbers 1-10 in this column). In second column you can put number of students who concluded they needed those number of colours.

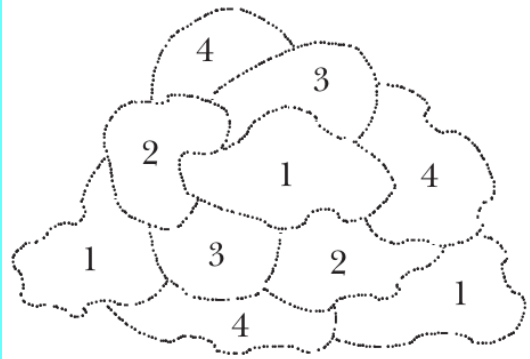


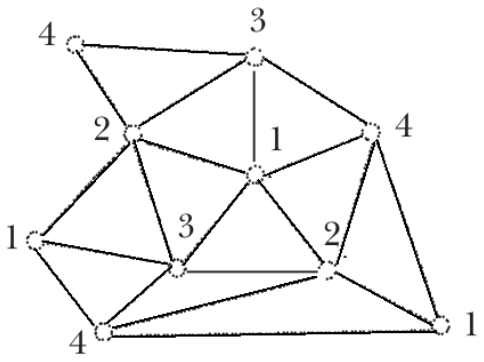
Probe them to think about

* Whether it will be possible to have a five colourable map?
* Is there any relation between previous problem?
* Can also give them Kerela map and ask to convert in graph. As the districts will be vertices and if they are sharing boundaries then there will be an edge joining that two vertices.

Can you convert this map into graph?

This is one of the answer for the given map. Students may get other representations.





First of all, ask the students why are these representations are the same? It may happen that they say no. Ask them why?
Give them hints like,

- Are the number of regions same? (which actually means are the number of vertices in graph same?)
- Are all their connections the same? (which actually means, are the vertices have same degrees?)
- Is number of colours used to code the graphs same?

Q. What do you think how many colours you will need to colour the vertices? _____

Task: 3 Time-table problems

A school is planing to start Vigyan Pratibha sessions for their Class 8 students. There are four different themes of subjects.

This is not easy! Because some students are taking interest in several subjects, and a student can attend only one session during a particular time slot. The school wants to avoid all conflicts. Of course, you can make such a schedule by having every sessions in a different time slot, but then you would need four periods in current timetable for the four Vigyan Pratibha themes, and the session would run all the day!

The teachers decided that sessions can be categorized according to 4 themes which is, Mathematics, Physics, Chemistry and Biology. The school asks the participating students to pick maximum two of the themes. After collecting choices of all students, teachers saw following different combinations:

Mathematics – Physics, Mathematics – Biology, Physics – Chemistry and Biology – Chemistry.

What is the minimum number of periods that would be allotted for these sessions so that every student can attend the sessions of their choice. Time-table came and on the very first day there were two parallel class of Physics and chemistry. Shama and two of her friends Rose and Amar chose Physics and Chemistry combination, but they couldn't attend the Physics class and Shama pointed out this to teacher immediately. Hence teacher has to make timetable again.

Can you help the teacher to arrange the timetable?

Let the students try their strategies and guess. (Give them 5-7 minutes time) Once they share ideas, you can tell them that "We can also solve this problem with a technique we call as map colouring."

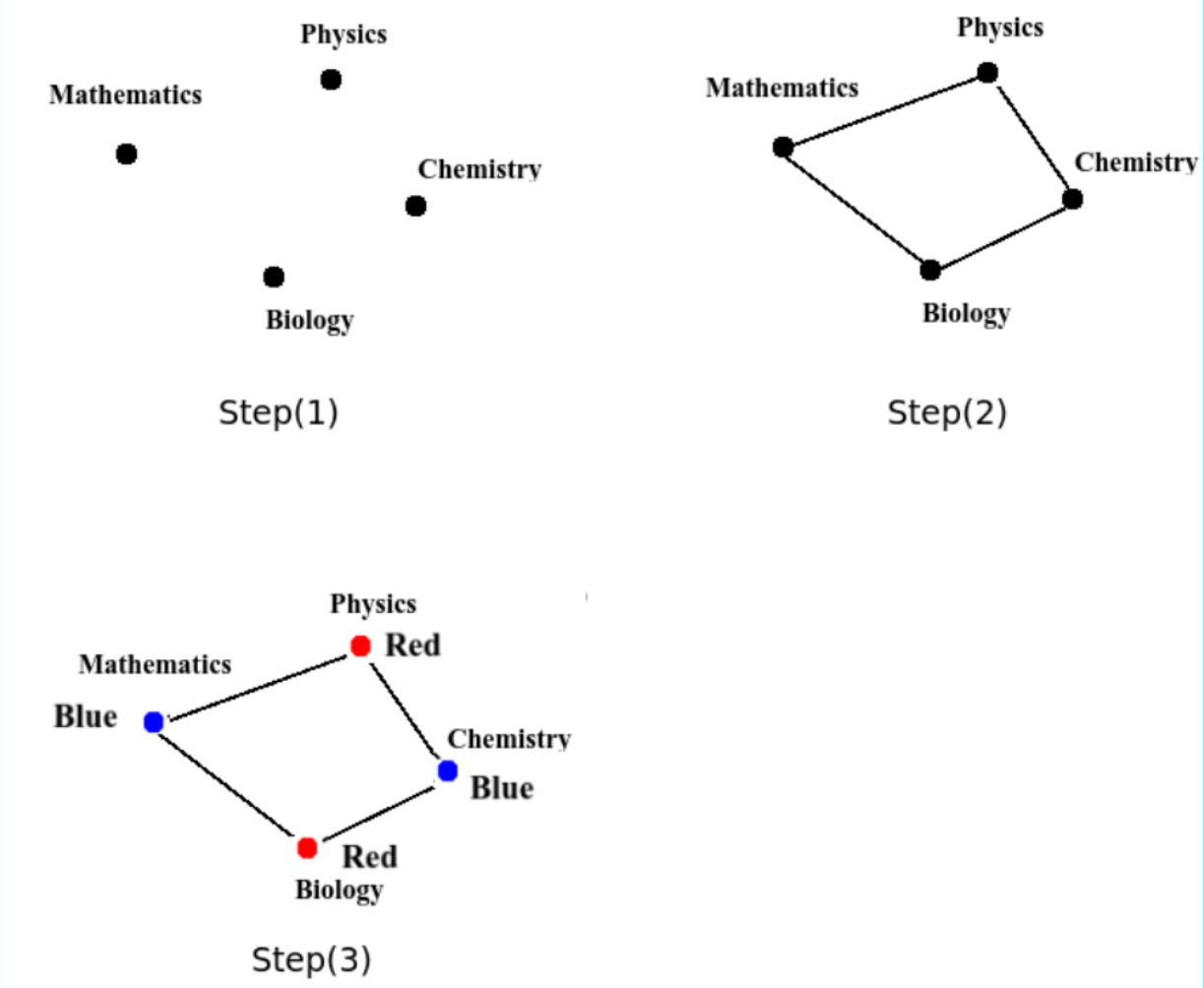
Give information that you want to adjust these problem solving sessions into the current timetable. Because students do have a feeling that why can't they arrange 4 periods in whole week. Discuss with them why you want minimum periods to adjust these sessions. Ask the students to observe the combinations once again. Bring to their notice that Mathematics and Chemistry are not a part of any combination. So they can have sessions simultaneous.

For example:
Mathematics – Physics
Mathematics – Biology

Physics – Chemistry
Biology – Chemistry

Encourage them to continue on these lines.

(1) We have four subjects. Let us put these subjects as four points (vertices of a graph)
(2) Connect any two subjects which have been chosen as a combination by any student by an edge; e.g. Mathematics and Physics are a part of a combination which is chosen, so we will connect Mathematics and Physics with an edge. Do the same for others.
(3) Now let us start colouring. We start by colouring Mathematics 'Blue'. Then, we cannot colour Physics blue because Mathematics and Physics are connected. So let us colour Physics 'Red'. Now, Chemistry is connected to Physics but not Mathematics. Thus, we can colour Chemistry blue. Now, Biology is connected to both Mathematics and Chemistry. Hence we can not colour it either blue, but Biology is not connected to Physics and hence we can colour Biology red.



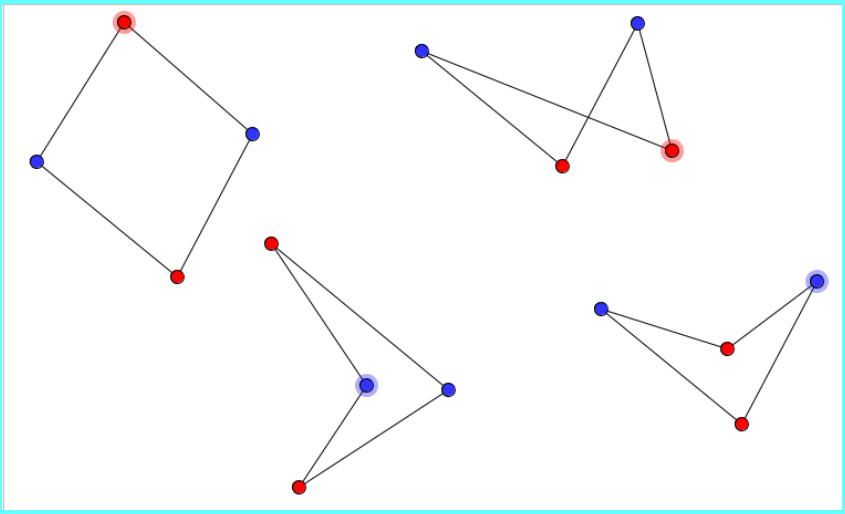
(4) Now we can all subjects with same colour to same period. As we needed only two colours to mark all the vertices, we just need two periods per week in the timetable to run these problem solving sessions.

Discuss with the students what changes in the combinations or the subjects will increase or decrease the periods needed. If we add a new subject, will it always increase the periods needed. What if we add a new combination? What will happen then?

Try to list the variations of students' representation. Some of the variations in above figures are shown below.

Teacher might have to explain here that how all these representation are same?

What matter is number of vertices and their degree. If that is equal than the representations has to be equal.



They might come up with some interesting ideas, and say that graph representation is not needed to solve this problem. Which is correct, to solve this problem graph is not the only way. Encourage to present their ideas in classroom. Also, explain what will they do if the data is use?

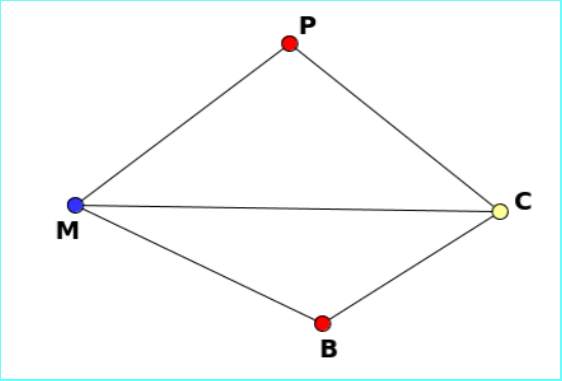
What if...

(I) There are a lot of children who want Chemistry and Mathematics both. The school accepted it and now the new combinations were:

Mathematics – Physics, **Mathematics – Biology**, **Physics – Chemistry**, **Biology – Chemistry** and **Mathematics – Chemistry**

Then what will be the minimum number of periods that would be allotted for these sessions so that every student can attend the sessions both the subjects of their choice?

Now because of Mathematics – Chemistry combination we can't have a same period allotted for both the subject. Which means we have to increase one more period in the timetable as students have selected the combination.



Here, 3 coloring indicate the minimum number required to arrange these combination classes.
Here students might note that there is a triangle formed and hence 3 different colours will be needed.

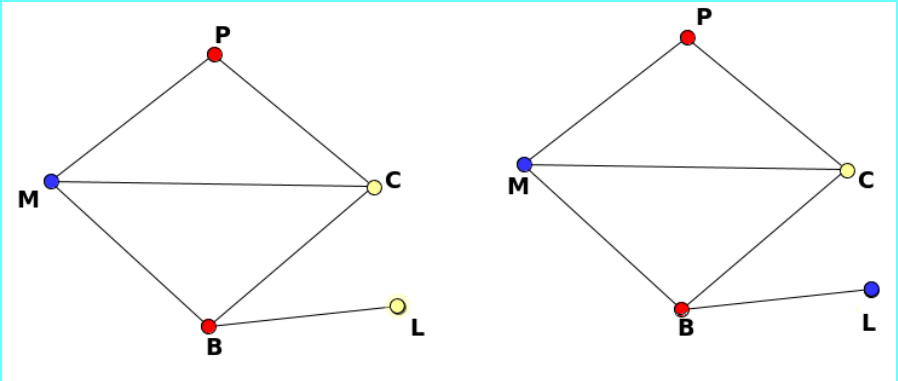
Now it's your turn,

What if... In the new term, the school decided to add another subject, Local Context to the list. So the new combinations were:

Mathematics – Physics, **Mathematics – Biology**, **Physics – Chemistry**, **Biology – Chemistry** , **Mathematics – Chemistry and Biology – Local Context.**

Now, what is the minimum number of periods that would be allotted for these sessions so that every student can attend the sessions for subjects of their choice?

Hence there are two possible answer which are shown below.



You can also explore, why this has happened? What is the difference between earlier question and this one?

Task 4: Real life situation

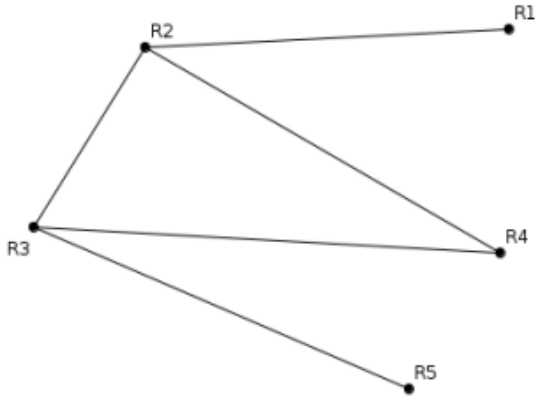
Have you heard any FM channels on the radio? Can you name some of these channels? In Mumbai, All India Radio (AIR) has several FM channels. Usually, some number is used to on dial of any FM radio to tune to any FM channel. e.g AIR's FM Gold 100.1, FM Rainbow 107.1 and FM Vividha Bharati 102.8. Do you know what these numbers mean? If you don't know the answer, discuss with your friends and teachers.

Have a discussion in the class about radio frequencies and how different stations run at different frequencies. Then ask them, "What will happen if two radio stations in neighbouring cities run (say Mumbai and Pune) at the same frequency?" If students struggle to come up with an answer, prompt them to think what will happen at midpoint of the two cities (say Lonavala). If some students answer saying that there will disturbance / the radio handset will get garbled mix of both stations, then ask them to tell the class why. When radio handset receives two signals at the same frequency, the listener would not be able to hear anything clearly.

Now let us attempt the problem of assigning broadcast frequencies to different radio stations. If we have too many different frequencies, then our handsets should be equipped to accepts frequencies from a wider range (bandwidth), which makes them more expensive. Thus, it is in our interest to allot as fewer frequencies as possible to the FM channels. At the same time, we must ensure that neighbouring radio stations should not get same frequency. What do we do? Again, let us start with a simple example:

Q. Let us start with say 5 radio stations R1, R2, R3, R4 and R5. Among these, R1 and R2 are neighbours. R2 is also a neighbour of R3 as well as R4. Further, R3 is a neighbour of R5. Given these conditions, how many minimum frequencies would be needed to ensure that there is no interference?

Radio Station	Neighbouring Radio Stations
R1	R2
R2	R1, R3, R4
R3	R2, R4, R5
R4	R2,R3
R5	R3



If we put the radio station as vertices and join the neighbouring stations with edges, the picture would look something like this.

Can you colour these vertices like previous task? In this case, number of colours would mean number of frequencies needed.

This should be fairly intuitive at this point. Three colours would be sufficient for this graph.

Q Can you construct a new situations and give your partner/friend to determine the minimum number of bandwidths required

What do you think what was common in all these activities? In all the activities, you tried to represent a given situation; maps, kingdoms, time-tables or radio stations using graphs and solved the problems using the idea of colouring.
This process of representation of real-life problems into a mathematical models (in these cases a graph) is called as mathematical modeling.