

Euclid's Game

Introduction

Playing games is a lot of fun. Today you are going to play a game that involves numbers, and based on these numbers you will find out a way to win the game, always!

Materials Required: Blackboard, chalk and sheets of paper

Task 1: Play the Euclid's Game

1. This is a two-player game.
2. The players take turns to write numbers, according to certain rules, till such time that it is no longer possible to write any new number.
3. The person who writes the last possible number is the winner.
4. The rules of the game are as follows:
 - You can decide who plays first.
 - The first player, say Player 1, writes down a number that is between 1 and 100, including both. Let's call this number 'A'.
 - The second player, say Player 2 can write down another number of his/her choice. Let's call this number 'B'.
 - Now, the first player will write the number $(A - B)$ or $(B - A)$, whichever is positive. Let's call this number 'C'.
 - Next, it is the second player's turn. He/she has a choice. He/She can either write the difference between C and A or the difference between C and B. However, if one of these differences is already written in the list (it is equal to A or B or C) then it cannot be written again.
 - Similarly, in subsequent turns, the players take turns to write a number which is the difference between any two numbers written in the list and is not already present in the list.
 - The game ends when it is not possible to write any new number.
 - The person who wrote the last number will be the winner.

Let us look at a sample run of the game.

- Suppose, the first player writes 12. The second player has 99 choices to choose his/her number. (as the upper limit is 100).
- Suppose, the second player chooses 16, then the first player can only write 4, i.e. the difference between 16 and 12.
- The next player then writes 8, the difference between 12 and 4. Note that the player could not have written the difference between 16 and 4, as 12 is already part of the sequences.
- Now there is no possibility of writing new numbers, so the game ends with the numbers 4, 8, 12, and 16 [with the game order 12, 16, 4, 8].
- There are 4 numbers in all, in the sequence, and here the second player is the winner, as he/she wrote the last number 8.

Play this game with your partner multiple times. Try to make sense of how the game progresses. For every game you play, record your observations in the table below. For the last column, where you record the winner, mention whether Player 1 (who chose the first number) won or Player 2 (who chose the second number) won.

Task 3: Proving that the Strategy Works

The surprising facts that came out of the activity are as follows:

Observation 1: The smallest number of the sequence is the GCD of the initial pair of numbers.

Observation 2: The sequence is made of the multiples of the smallest number.

1. Can you figure out why this happened with every pair of numbers?

Let us look at the two observations.

- Observation 1 above implies the following:
 - a) There exists the smallest number of the sequence.
 - b) This smallest number divides both the initial numbers.
 - c) The smallest number is not just any common divisor, but the greatest common divisor of the initial numbers.
- Observation 2 entails the following:
 - a) All the numbers in the sequence are multiples of the smallest number,
 - b) All the multiples of the smallest number appear in the sequence up to the largest number.

2. We need to prove or justify these observations. Can you think about the ways of doing the same?

Points to Ponder

- If we were playing the game this way, will all the multiples of s be part of the sequence?
- Do all pairs of numbers allow for a winning strategy? If no what kinds of numbers allow for a winning strategy?
- What happens if you allow for first 3 numbers to be random? Say make it a 3 player game?

References

- Euclid's Game: <https://www.cut-the-knot.org/blue/EuclidAlg.shtml>
- The optimal strategy in Euclid's game: <https://math.stackexchange.com/questions/754461/optimal-strategy-in-euclids-game>