# Exploring Irrational Numbers

## Introduction

### Part 1:

In this learning unit we will find out something about the history of irrational numbers and explore the world of irrational numbers through right angle triangles. Through various construction exercises, we will get familiarized with operations on irrational numbers.

Consider two ribbons, Ribbon 1 and Ribbon 2 of different lengths measured in some common units. Now I want to measure the lengths of both the ribbons using the same stick. The condition is that the the length of both the ribbons should be whole number multiples of the length of the stick, as there are no markings on the stick.

Solve some questions based on the above situation.

1) Length of Ribbon 1 = 3 cm
   Length of Ribbon 2 = 9 cm

What will be the length of the stick you will use to measure these two ribbons?
Length of the stick = ________ cm

Discuss your answers with your friends.

2) Length of Ribbon 1 = 4 cm
   Length of Ribbon 2 = 6 cm

What will be the length of the stick you will use to measure these two ribbons?

Length of the stick = ________ cm

Discuss your answers with your friends.

3) Length of Ribbon 1 = 3 cm
   Length of Ribbon 2 = 4 cm

What can be the length of the stick you will use to measure these two ribbons?

Length of the stick = ________ cm

Discuss your answers with your friends.

Now instead of the ribbons let us look at line segments. Let AB and CD be two line segments of lengths *r* and *s*. And when we compare the shorter length, *r* to the longer length, *s*, if we find that *r* fits exactly a whole number of times into *s*. Then we say that *r* is a measure of *s*, and *s* is a multiple of *r*.

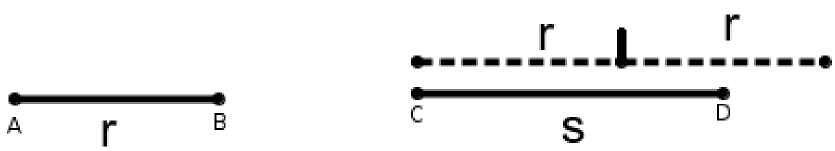
For example, length of AB is *r* and length of CD is *s*,


Then, we find that *r* fits exactly 3 times into *s* that is *s* = ……×*r* and *s/r* = __/__ (Fill in the blanks)

So we choose *r* as the unit of measurement then length of *r* is 1 unit and the length of *s* is __ units.

Of course, it is possible that *r* does not fit exactly a whole number of times into *s* but it may be possible to find a smaller length *y* that fits a whole number of times into *r* and *s* both. Then we say that *y* is a common measure of *r* and *s*.
Notice that, then the ratio of *s/r* may be expressed as a ratio of whole numbers.

For example, length of AB is *r* and length of CD is *s*.




Then *r* does not fit exactly a whole number of times into *s*. But can we find *y* such that *y* fits exactly whole number of times into both *r* and *s*?
Let us see how. Consider a line segment EF whose length is *y*,


So *y* fits into *r* exactly ___ times and *s* exactly ___ times.

Is *y* a common measure of *r* and *s*?

___________ (Yes/No)

And, *r* = __ *y* and *s* = __ *y*.

**Task 1:**

Look at the following numbers and check if one is the measure of another. If not, then try to find a common measure. Compare your answers with your friends' answers.

| Pair No. | *r* | *s* | Is *r* a measure of *s*? | Common measure (*y*) |
|---|---|---|---|---|
| 1 | 2 cm | 6 cm | Yes | 2 cm, 1 cm or 0.5 cm |
| 2 | 3 cm | 12 cm | | |
| 3 | 3 cm | 5 cm | | |
| 4 | 4 cm | 18 cm | | |
| 5 | 15 cm | 36 cm | | |

In all the pairs was *r* a measure of *s*? ______________________

**Task 2:**

Fill in the following table

| Pair No. | Length 1 (*L1*) | Length 2 (*L2*) | Is *L1* a measure of *L2*? | Is there a common measure? | Common measure |
|---|---|---|---|---|---|
| 1 | 1 cm | 4.2 cm | | | |
| 2 | 2 cm | 3.5 cm | | | |
| 3 | 2.5 cm | 6 cm | | | |
| 4 | 1/3 cm | 1/2 cm | | | |
| 5 | 1/6 cm | 1/4 cm | | | |
| 6 | 5/6 cm | ¾ cm | | | |

Compare your results with your friends.

Given two line segments if one is a measure of another or there is a common measure for both of them then these numbers (lengths of the line segments) are called ***commensurable***.

So in short, two line segments are called commensurable if you could find a smaller line segment that could be used as a "unit" or "ruler" (measure or a common measure) with which you can measure both the given line segments.

**Task 3:**

Take any two fractions (non-unit fractions) of your choice and try to find a common measure for them.

**Task 4:**

Given two line segments of lengths $\frac{p}{q}$ units and $\frac{n}{m}$ units, can you find a common measure for them?

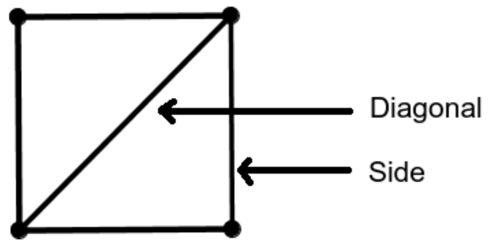
You have proved a very important result. This result shows that any rational number is commensurable with 1 unit.

You must have heard about the Pythagoras Theorem. The Pythagoras theorem is named after a famous mathematician and philosopher Pythagoras. Pythagoras gained his famous status by founding a group, the Brotherhood of Pythagoreans, which was devoted to the study of mathematics.

'*The Pythagoreans believed that given any two line segments, one is a measure of the other or you can always find a common measure for them. That is, for any two line segments either one line segment is a measure of the other or there was a third line segment which is a common measure of both the original line segments*.'

This belief was shattered when they found a pair of lengths that did NOT have a common measure. The two lengths were the side of a square and its diagonal.

Let us find out more about these lengths.




Let the length of the side be 1 unit. Then what can you say about the length of the diagonal, *s*?

Step 1:

$s^2 = 1^2 + 1^2 = 2$ .............Pythagoras Theorem

Let us assume that 1 and *s* have a common measure then there exists a number y such that $s = n \times y$ and $1 = m \times y$ where *n* and *m* are whole numbers. Then we get that, $\frac{s}{1} = \frac{n}{m}$ ------- (1)

Let us assume that *y* is largest possible common measure of 1 and s.

(Notice that this is a fair assumption, as the common measure cannot be greater than the smaller of the two lengths.)

Actually the common measure has to be strictly less than 1 because $s^2 = 2$ so

*s* has to be between 1 and 2.

So, 1 = *my* and *s* = *ny*

Pythagoras theorem says that,
$s^2 = 1^2 + 1^2 = 2$

But $\frac{s^2}{1} = \frac{n^2}{m^2}$ ------------ Squaring (1)
So,
$s^2 = \frac{n^2}{m^2}$
Then, $n^2 = s^2 \times m^2 = 2m^2$
So we can say that *n* is an even number. Why?

So *n* = 2*k*, *k* is a whole number.
$n^2$ = ____ $k^2$ ..... (Fill in the blank)
But, $n^2 = 2m^2$

So, $2m^2$ = _______ ..... (Fill in the blank)
or $m^2$ = _______..... (Fill in the blank)

This tells us that *m* is also an even number. Why?

_______________________________________________

So, $m = 2m'$ and $n = 2n'$.
Recall that, $1 = my$ and $s = ny$

So, we get that, $1 = 2m'y$ and $s =$ _____
That is, $1 = m' \times 2y$ and $s = n' \times 2$ __

So, _____is also a common measure for 1 and $s$. (Fill in the blanks)

Now, $2y$ is obviously greater than the greatest common measure, $y$.
This actually contradicts what we had started with.

We were led into this absurdity as a result of assuming that the length $s$ may be expressed as a ratio of whole numbers. So our assumption must be wrong.

So diagonal of square and its side never have a common measure or are always incommensurable.

This particular result actually had a huge impact on how people did mathematics then. Until then geometry and arithmetic were looked at as one subject but because the arithmetic then was not advanced enough to provide a way for expressing by numbers the ratio of two lengths that do not have a common measure, people started looking at geometry and arithmetic as different subjects.

This proof shows the existence of line segments whose lengths could not be expressed by then existing numbers . It is said that because of this, arithmetic and geometry went their separate ways for more than two thousand years until an improved arithmetic made it possible to express by numbers the ratio of two lengths that do not have a common measure.

### Part 2:

In the coming tasks, we will geometrically construct line segments of different lengths which are irrational numbers.

### Task 5:

Draw a right angle triangle such that two of its sides are of unit length. What can you say about the length of its hypotenuse?

### Task 6:

Using the hypotenuse obtained in Task 5 as one leg and one leg with unit length draw a right angle and complete the triangle. What is the length of hypotenuse of the new triangle?

**Task 7:**

Continue this process for minimum 5 steps.

**Task 8:**

Draw a similar spiral starting with one of the sides of the triangle as 6 units and the other as 1 unit instead of both sides of 1 unit. Continue for a minimum 5 steps. What can you say about the length of the last line segment you constructed?

**Task 9:**

How will you construct line segments whose length is equal to the following numbers? Give justifications as to why your answers are correct.

1) $\sqrt{32}$

2) $\sqrt{40}$

3) $\sqrt{50}$

4) $\sqrt{63}$

**Part 3**

**Task 10:**

Draw a right angle triangle such that its two sides which are at right angles are $\sqrt{2}$ and $\sqrt{3}$. ( $\sqrt{2}$ and $\sqrt{3}$ can be drawn using the techniques you figured out in the first part of this learning unit.) What is the length of its hypotenuse?

**Task 11:**

If you draw a right angle triangle such that the two right angle sides are $\sqrt{n}$ and $\sqrt{m}$, what is the length of the hypotenuse?

**Task 12:**

Can you draw a right angle triangle whose all sides are integers? Draw at least two different right angle triangles having this property. What kind of numbers did you get as side-lengths?

**Task 13:**

Can you draw a right angle triangle such that the two right angle sides are integers and the hypotenuse is an irrational numbers? Draw at least two different right angle triangles having this property. What kind of numbers did you get as side-lengths?

**Task 14:**

Can you draw a right angle triangle such that the hypotenuse is an integer and one of the other sides is also an integer and the third side is an irrational number. Draw at least two different right angle triangles having this property. What kind of numbers did you get?

**Task 15:**

Can you draw a right angle triangle such that the two right angle sides are irrational numbers and the hypotenuse is an integer? Draw at least two different right angle triangles having this property. What kind of numbers did you get?

**Task 16:**

Can you draw a right angle triangle such that the hypotenuse is an irrational number and one of the other sides is also an irrational number and the third side is an integer. Draw at least two different right angle triangle having this property. What kind of numbers did you get?

You can ask students to explore many such examples. Try to encourage them to figure out a strategy to get such examples.

**Task 17:**

Can you draw a right angle triangle whose all sides are irrational numbers? Draw at least two different right angle triangles having this property. What kind of numbers did you get?

## References

The Dangerous Ratio (https://nrich.maths.org/2671)
To Infinity and Beyond: A Cultural History of the Infinite
Dantzig, Tobias (1954). Number, the Language of Science. New York, Free Press.